Bachelor of Science (Major) Examination, 2024

(First Year, First Semester)

Mathematics

Paper : CORE – 02

(Geometry and Linear Algebra)

Time : 2 *Hours*

Full Marks : 40

Use separate answer scripts for each Part.

PART—I (20 Marks)

Answer ***any five*** questions from the following : (4×5)

1. A sphere of radius $2k$ passes through the origin and meets the axes in A, B and C respectively. Show that the locus of the centroid of the tetrahedron $OABC$ is the sphere $(x^2+y^2+z^2)=k^2$.

2. Find the equation of the cone with vertex at origin, which passes through the curve of intersection of plane $lx+my+nz=p$ and $ax^2+by^2+cz^2=0$.

3. PSP' is a focal chord of the conic. Prove that the angle between tangents at P and P' is $\tan^{-1}\left(\frac{2e\sin\alpha}{1-e^2}\right)$, where α is the angle between the chord and the major axis.

4. Obtain the equation of the cylinder, whose generators intersect the plane curve $ax^2 + by^2 = 1, z = 0$ and are parallel to the straight line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$.

5. Prove that the length of the common chord of circles $(x - a)^2 + (y - b)^2 = c^2$ and $(x - b)^2 + (y - a)^2 = c^2$ is $\sqrt{4c^2 - 2(a-b)^2}$.

6. Show that the area enclosed by the curve in which the plane $z = h$ cuts the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is $\pi ab\left(1 - \frac{h^2}{c^2}\right)$.

PART—II (20 Marks)

Let $\mathbb{R}$ denote the field of all real numbers.

Answer *any four* questions : 4×5

1. Define a *subspace* of a vector space. Determine whether S is a subspace of $\mathbb{R}^5$, where

$$S = \left\{(a_1, a_2, a_3, a_4, a_5) \in \mathbb{R}^5 \mid a_1^2 + a_3^2 = 0, 2a_2 + 3a_5 = 5a_4\right\}.$$

Find a basis and the dimension of S over $\mathbb{R}$ if S is a subspace of $\mathbb{R}^5$. 1+4

2. Solve the following system of linear equations by Gaussian elimination process :

$$\begin{aligned} x_1 - 2x_2 + 2x_4 - 6x_5 &= 4 \\ 2x_1 - 4x_2 + 2x_3 + 4x_5 &= 6 \\ x_1 - 2x_2 + 3x_3 - 3x_4 + 10x_5 &= 16 \end{aligned}$$ 5

3. Define a *basis* of a vector space. Find a basis of $\mathbb{R}^5$ that contains $\{(1, 0, -4, 3, 5), (-2, 1, 2, 2, -3)\}$. 1+4

4. Define V be a finite dimensional vector space over $\mathbb{R}$. Let $T: V \to V$ be a linear transformation. Prove that T is one-to-one if and only if T is onto. 5

5. Let $T: \mathbb{R}^4 \to \mathbb{R}^2$ be the linear transformation defined by
$T(x, y, z, t) = (x + 5y - 3z + t, 4z - 5t)$.
Find the matrix representation of T with respect to the ordered bases
$\{(1, -1, 1, 0), (0, 2, -2, 1), (1, 1, 1, 1), (3, 2, 1, 0)\}$ and $\{(2, 3), (5, 7)\}$ of $\mathbb{R}^4$ and $\mathbb{R}^2$ respectively. 5

6. Find eigenvalues and corresponding eigen-spaces of the matrix and determine whether it is diagonalizable.

$$\begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix}$$

1+2+2

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Ex/SC/MATH/UG/MAJOR/TH/12/103/2024

Bachelor of Science Examination, 2024

(1^{st} Year, 2^{nd} Semester)

Mathematics (Honours)

Paper : MAJOR - 03

(Algebra)

Time : 2 *Hours* | *Full Marks* : 40

PART—I

(Marks : 28)

CO 1 : Answer *any* **one** question :

1. *(a)* Define a group. Verify whether the following set G of real matrices forms a group under usual matrix multiplication :

$$G = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : A^T A = I_2 \right\}$$

where I_2 is the identity matrix of order 2. 4

(b) Define a Boolean ring. Let R be a Boolean ring. Show that $2x = 0$ and $xy = yx$ for all $x, y \in R$. 3

2. (a) Define the order $o(a)$ of an element a of a group. Let G be a group and $a, b \in G$ be such that $b \neq e, a^3 = e$ and $aba^{-1} = b^2$. Show that $o(b) = 7$. 4

(b) Define an integral domain. Prove that a finite integral domain is a field. 3

CO 2 : Answer *any* **one** question :

3. (a) Define a subgroup. Prove that an infinite group has infinitely many subgroups. 4

(b) Prove that any finite subgroup of the group of nonzero complex numbers (with usual multiplication) is a cyclic group. 3

4. (a) State and prove Lagrange's theorem for finite groups. 5

(b) Let G be a finite group and $a \in G$. Show that $\circ(a)$ divides $\circ(G)$. 2

CO 3 : Answer *any* **one** question :

5. (a) Define a normal subgroup of a group. Let H be a subgroup of a group G. If for each $a \in G$, there exists $b \in G$ such that $aH = Hb$, then prove that H is a normal subgroup of G. 4

(*b*) Prove that the quotient group $(6\mathbb{Z}/30\mathbb{Z}, +)$ is isomorphic to the group $(\mathbb{Z}_5, +)$. 3

6. (*a*) Define the kernel of a homomorphism of groups. Prove that every normal subgroup of a group G is a kernel of some homomorphism defined on G. 4

(*b*) How many homomorphisms can be defined from the symmetric group S_3 to the group $(\mathbb{Z}_6, +)$? Justify. 3

CO 4 : Answer *any* **one** question :

7. (*a*) Let G be a finite commutative group and $a, b \in G$ such that $o(a) = 30$ and $o(b) = 40$. Prove that G has an element of order 120. 4

(*b*) Prove that every commutative group of order 15 is a cyclic group. 3

8. (*a*) Prove that $\mathbb{Z}_m \times \mathbb{Z}_n \cong \mathbb{Z}_{mn}$ if and only if $\gcd(m, n) = 1$ for all $m, n \in \mathbb{N}$. 4

(*b*) Prove that every even permutation of S_n is a finite product of 3 cycles. 3

PART—II

(Marks : 12)

Answer *any* **two** questions from (**1** to **3**) and *any* **one** from (**4** to **5**) :

1. Show that the roots of the equation

$$\frac{1}{x+a_1}+\frac{1}{x+a_2}+\cdots+\frac{1}{x+a_n}=\frac{1}{x+b}$$

are all real, where $a_i, b \in \mathbb{R}^+$ with $b > a_i \; \forall i$. 4 **[CO1]**

2. If $a, b, c \in \mathbb{R}^+$, then show that

$$\frac{9}{a+b+c} \le \frac{2}{a+b}+\frac{2}{b+c}+\frac{2}{c+a} \le \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$ 4 **[CO2]**

3. *(a)* Find the nature of the roots of the equation $x^6 - 1 = 0$.

(b) If the roots of the equation $x^3 + px^2 + qx + r = 0$ $(r \neq 0)$ are α, β, γ, then find the equation whose roots are $\frac{\alpha+\beta}{\gamma}, \frac{\beta+\gamma}{\alpha}, \frac{\gamma+\alpha}{\beta}$. 2+2 **[CO1]**

4. Solve the equation $x^3 - 6x + 4 = 0$ by Cardan's method.

4 **[CO2]**

5. Solve the equation $x^4 - 6x^2 + 16x - 15 = 0$ by Ferrari's method. 4 **[CO2]**

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