Bachelor of Science Examination, 2024

(1st Year, 1st Semester)

Mathematics

Paper : MAJOR – 101

(Real Analysis)

Time : 2 *Hours* | *Full Marks* : 40

Use a separate Answer-Script for each Part.

PART—I

(Marks : 20)

Answer ***any five*** questions : (4×5=20)

1. Show that the set [0, 1] is uncountable. 4

2. Let F be an Archimedean ordered field. Show that if F satisfies least upper bound property then F has Cantor's nested interval property. 4

3. Show that interior of a set is the largest open set contained in the set. Find the derived set of $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{Q}^c$. 2+2

4. Prove that in $\mathbb{R}$ finite intersection of open sets is open. Give an example to show that arbitrary intersection of open sets may not be open. 2+2

5. Find the closure of the set $\{m + n\sqrt{2} : m, n \in \mathbb{Z}\}$. 4

6. Prove that every closed and bounded set in $\mathbb{R}$ is compact. 4

7. Show that an element x_0 is a limit point of a set S if and only if there exists a sequence $\{x_n\}$ of elements from $S \setminus \{x_0\}$ converging to x_0. 4

8. Prove that the set $S = \{x \in \mathbb{Q} : 2 < x^2 < 3\}$ is both closed and open in $\mathbb{Q}$. Justify whether the set S is compact or not. 2+2

PART—II

(Marks : 20)

Answer ***any four*** questions :

1. *(a)* If the subsequences $\{x_{3n-2}\}, \{x_{3n-1}\}$ and $\{x_{3n}\}$ of a sequence $\{x_n\}$ converge to the same limit ℓ, then prove that $\{x_n\}$ converges to ℓ.

   (b) Prove that $n^{1/n} \to 1$ as $n \to \infty$ 3+2

2. Define nests of intervals. For any nest of closed intervals $\{[a_n, b_n]\}$, prove that there exists a unique real number x such that $x \in [a_n, b_n] \, \forall n$. 1+4

3. Prove that $\left\{\left(1+\frac{1}{n}\right)^n\right\}$ is convergent and

~~$2 \leq \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \leq 3.$~~ 5

$2 < \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n < 3.$

4. (a) Prove that the sequence $\{u_n\}$ defined by $u_1 = \sqrt{7}$ and $u_{n+1} = \sqrt{7u_n} \, \forall n \geq 1$, converges to 7.

   (b) If $\{x_n\}$ is a bounded sequence and $\{y_n\}$ converges to 0, then prove that $\{x_n y_n\}$ converges to 0. 3+2

5. (a) Let $\sum u_n$ and $\sum v_n$ be two series of positive real numbers and $\lim_{n\to\infty} \frac{u_n}{v_n} = \ell$. If $\ell \neq 0$, then prove that $\sum u_n$ and $\sum v_n$ converge or diverge together.

   (b) If $\sum u_n$ is a convergent series of positive real numbers, then prove that $\sum \frac{u_n}{s+u_n}$ is convergent for any non-zero real number $s > 0$. 3+2

6. (a) Test the convergence of the following series :

$$\sum \frac{a^n}{n}, \ a > 0$$

(b) Test the convergence of the following series :

$$\frac{1+2}{2^3} + \frac{1+2+3}{3^3} + \frac{1+2+3+4}{4^3} + \cdots$$

[3+2]

7. Let $\sum a_n$ be a convergent series of positive real numbers.

(i) Prove that the sequence of nth partial sum of the series is bounded above.

(ii) If $\{a_n\}$ is monotonically decreasing sequence, then prove that $n a_n \to 0$ as $n \to \alpha$. [2+3]

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Ex/SC/MATH/UG/MAJOR/TH/12/104/2024

Bachelor of Science Examination, 2024

(1^{st} Year, 2^{nd} Semester)

Mathematics

Paper : MAJOR - 104

(Theory of Real Functions)

Time : *Two Hours* *Full Marks* : 40

Use separate sheet for each Part.
Symbols and notations have their usual meanings.

PART—I (Marks : 20)

Answer *any* **four** questions.

1. *(i)* Let $f:\mathbb{R}\to\mathbb{R}$ be continuous on $\mathbb{R}$ and $f(x_0)>c$ for some x_0 in $\mathbb{R}$. Then show that there exists a $\delta>0$ such that $f(x)>c\ \forall x\in(x_0-\delta, x_0+\delta)$.

   (ii) Let $S=\{x\in\mathbb{R}: e^x+\sin x>1\}$. Justify whether S is open in $\mathbb{R}$ or not? 3+2

2. *(i)* Let $f:(a,b)\to\mathbb{R}$ be continuous. Then show that f is uniformly continuous iff $\lim\limits_{x\to a+} f(x)$ and $\lim\limits_{x\to b-} f(x)$ exist finitely.

(ii) Hence or otherwise show that $f:(0,1)\to\mathbb{R}$ defined by $f(x)=1/x$ is not uniformly continuous. 4+1

3. Let $f:\mathbb{R}\to\mathbb{R}$ be such that f satisfies Cauchy functional equation

$$f(x+y)=f(x)+f(y)\ \forall x, y\in R$$

If f is continuous at a point $x_0\in\mathbb{R}$, then show that f is continuous on $\mathbb{R}$ and there exists a constant c in $\mathbb{R}$ such that

$$f(x)=cx\ \forall x\in\mathbb{R}$$ 5

4. Let $X, Y\subset\mathbb{R}$ and $f:X\to Y$ be a function which is invertible. Let $x_0\in X$ and $f(x_0)=y_0$. If f is differentiable at x_0 and f^{-1} is continuous at y_0, $f'(x_0)\neq 0$, then prove that f^{-1} is differentiable at y_0 and

$$(f^{-1})'(y_0)=\frac{1}{f'(x_0)}$$ 5

5. Let $f:[a,b]\to\mathbb{R}$ be a function such that f is differentiable at every point of $[a, b]$. If $f'(a)<\gamma<f'(b)$, then show that there exists $c\in(a,b)$ such that $f'(c)=\gamma$. Hence or otherwise show that if f is differentiable on $[a, b]$, then f' cannot have any simple discontinuities. 4+1

6. (i) Show that $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=x^2$ is not uniformly continuous on $\mathbb{R}$.

(ii) Find the differential of the function $f(x)=x^3$ at the point $x=3$. 3+2

PART—II (Marks : 20)

Answer the following questions.

1. **[CO-1] :**

State and prove Taylor's theorem with Cauchy's form of remainder. 5

(OR)

Find $\underset{x\to 0}{\text{Lt}}\ (\cos x)^{\cot^2 x}$

Show that $\int_0^{\frac{\pi}{2}} \cos^n x \cos nx\, dx = \frac{\pi}{2^{n+1}} (n \in \mathbb{Z}_+)$. 2½+2½

2. **[CO-3] :**

Define convex function.

If $a, b, c > 0$ and $a+b+c=1$, then find the minimum value of $\left(a+\frac{1}{a}\right)^{10}+\left(b+\frac{1}{b}\right)^{10}+\left(c+\frac{1}{c}\right)^{10}$. 1+4

(OR)

Let I be an open interval and $f: I \to \mathbb{R}$ be such that $f''(x)$ exists on I. Then prove that f is a convex on I iff $f''(x) \geq 0\ \forall x \in \mathbb{I}$. 5

3. [CO-2] : Answer *any* **two** : 5×2=10

(a) If $y = e^{a\sin^{-1}x}$, then show that

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0$$

Find y_n (n even) when $x = 0$.

(b) Find all the asymptotes of the curve

$$(x+y)^2(x+2y+2) - x - 9y + 2 = 0$$

(c) Find the volume of the solid obtained by revolving the cardioid $r = a(1+\cos\theta)$ about the initial line.

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