Theory of Real Functions

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Unit - 1
Limits of functions (ε - δ approach), sequential criterion for limits, divergence criteria. Limit theorems, one sided limits. Infinite limits and limits at infinity. Continuous functions, sequential criterion for continuity and discontinuity. Algebra of continuous functions. Continuous functions on closed and bounded interval, intermediate value theorem, location of roots theorem, preservation of intervals theorem. Classification of discontinuity, discontinuity of monotonic functions. Uniform continuity, non-uniform continuity criteria, uniform continuity theorem on compact sets.

Unit - 2
Differentiability of a function at a point and in an interval, Caratheodory's theorem, algebra of differentiable functions. Relative extrema, interior extremum theorem. Rolle's theorem. Mean value theorem: Lagrange's mean value theorem, Cauchy's mean value theorem, Darboux's theorem on derivatives. Applications of mean value theorem to inequalities and approximation of polynomials.

Unit - 3
Taylor's theorem with Lagrange's form of remainder, Taylor's theorem with Cauchy's form of remainder and Young's form of remainder, application of Taylor's theorem to convex functions, Jensen's inequality, relative extrema. Taylor's series and Maclaurin's series expansions of exponential and trigonometric functions, $\ln(1+x)$, $\frac{1}{(ax+b)}$ and $(x+1)^n$. Application of Taylor's theorem to inequalities. L'Hospital's rule.

Unit - 4
Higher order derivatives, Leibnitz rule, concavity and inflection points, envelopes, asymptotes, curvature, curve tracing in cartesian coordinates. Reduction formulae, derivations and illustrations of reduction formulae, parametric equations, parameterizing a curve, arc length of a curve, arc length of parametric curves, area under a curve, area and volume of surface of revolution. [50]

Contents

§1 Limits and Continuity

Notes from the professor are appended in the following pages.

[1]

LIMIT AND CONTINUITY

PROF KALLOL PAUL

1. **Limit**

Definition 1.1. (Limit Point/Cluster Point)
Let $A \subseteq \mathbb{R}$. A point $c \in \mathbb{R}$ is a limit point (cluster point) of A if for every $\delta > 0$ there exists at least one point $x \in A,\ x \neq c$ such that $|x - c| < \delta$.

Theorem 1.2. *A number $c \in \mathbb{R}$ is a limit point of a subset A of $\mathbb{R}$ if and only if there exists a sequence $\{a_n\}$ in A such that $\lim_{n\to\infty}\{a_n\} = c$ and $a_n \neq c$ for all $n \in \mathbb{N}$.*

Definition 1.3. (Limit of a function at a point)
Let $A(\neq \emptyset) \subseteq \mathbb{R}$ and c be a limit point of A. Then a function $f : A \to \mathbb{R}$, is said to have a limit at c, if there exists a fixed real number L such that for any given real number $\epsilon > 0$, there exists a real number $\delta > 0$ (depending on both ϵ and the point c) such that

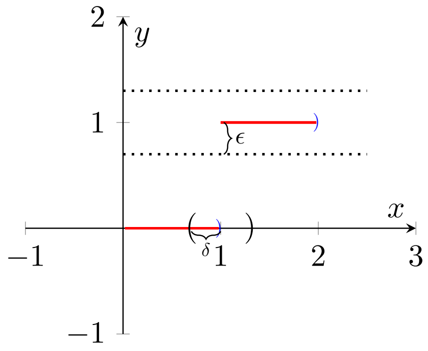
$$|f(x) - L| < \epsilon \text{ whenever } |x - c| < \delta \text{ and } x \in A.$$

We often write it as

$$\lim_{x\to c} f(x) = L.$$

Example 1.4. Let $f : (0, 2) \longrightarrow \mathbb{R}$ be defined by

$$\begin{aligned} f(x) &= 0,\ x < 1 \\ &= 1,\ 1 \geq x < 2 \end{aligned}$$



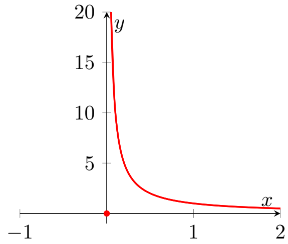
For any $\delta > 0$, there exists $x_1 \in (1 - \delta, 1 + \delta)$ such that $x_1 < 1$ and so $f(x_1) = 0$. So if we take $0 < \epsilon < 1$, then $|x - 1| < \delta \not\Rightarrow |f(x) - 1| < \epsilon$. Hence 1 is not the limit of f at 1.
Note that if we take $\epsilon > 1$, then there exists $\delta > 0$ such that $|x - 1| < \delta \Rightarrow |f(x) - 1| < \epsilon$.

[1]For any further readings please see books by Rudin, Apostol or Bartle and Scherbert.

Next, we give an example, for which there does not exist any $\delta > 0$, for any $\epsilon > 0$ such that $|x - c| < \delta \Rightarrow |f(x) - L| < \epsilon$.

Example 1.5. Let $f : (0, \infty) \longrightarrow \mathbb{R}$ be defined by

$$\begin{aligned} f(x) &= 0, \; x = 0 \\ &= \frac{1}{x}, \; x > 0. \end{aligned}$$

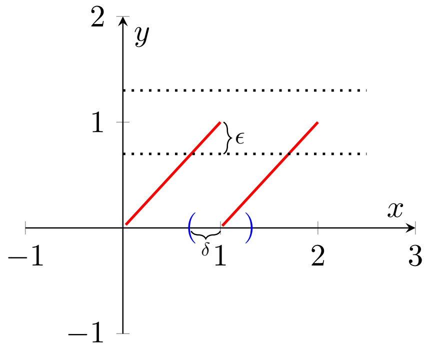


We check whether 0 is a limit of f at 0 or not. Let $\epsilon > 0$. If possible let there exist a $\delta > 0$ such that $|x - 0| < \delta \Rightarrow |f(x) - 0| < \epsilon$. By Archimedean property of $\mathbb{R}$, there exists $n_0 \in \mathbb{N}$ such that $\delta > \frac{1}{n_0}$. Let $n > \max\{n_0, \epsilon\}$, then $\frac{1}{n} \in (0, \delta)$ but $f(\frac{1}{n}) = n > \epsilon$, which is a contradiction. Thus for any $\epsilon > 0$ there does not exist any $\delta > 0$ such that $|x - 0| < \delta \Rightarrow |f(x) - 0| < \epsilon$.

Next, we give an example, where for each $\delta > 0$, there exists $\epsilon > 0$, such that $|x - c| < \delta \Rightarrow |f(x) - l| < \epsilon$, but the limit does not exist.

Example 1.6. Let $f : [0, 2] \longrightarrow \mathbb{R}$ be defined by

$$\begin{aligned} f(x) &= x, \; x \leq 1 \\ &= x - 1, \; 1 < x \leq 2 \end{aligned}$$



Here for each $\delta > 0$, there exists ϵ (take $\epsilon \geq 1$) such that $|x - 1| < \delta \Rightarrow |f(x) - 1| < \epsilon$. But for $\epsilon < 1$, there does not exists $\delta > 0$, such that $|x - 1| < \delta \Rightarrow |f(x) - 1| < \epsilon$. So at $x = 1$, the function $f(x)$ does not have a limit.

Theorem 1.7. *Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ and let c be a limit point of A. Then f can have only one limit at c.*

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Proof. Let $L, L' \in \mathbb{R}$ be the two limits of f at c. Now for given any $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \frac{\epsilon}{2}$ whenever $0 < |x - c| < \delta$ and $x \in A$. Also there exists $\delta' > 0$ such that $|f(x) - L'| < \frac{\epsilon}{2}$ whenever $0 < |x - c| < \delta'$ and $x \in A$. Let $\delta_0 := \min\{\delta, \delta'\}$. Then if $x \in A$ and $0 < |x - c| < \delta_0$, the Triangle Inequality implies that

$$|L - L'| = |L - f(x) + f(x) - L'| \leq |L - f(x)| + |f(x) - L'| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we conclude that $L - L' = 0$, i.e., $L = L'$. This complete the proof of the theorem. □

Theorem 1.8. *Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ and let c be a limit point of A. Then the following statements are equivalent:*
(i) $\lim_{x \to c} f(x) = L$.
(ii) *For every sequence $\{x_n\}$ in A that converges to c such that $x_n \neq c$ for all $n \in \mathbb{N}$, the sequence $\{f(x_n)\}$ converges to L.*

Proof. $(i) \Rightarrow (ii)$. Assume f has limit L at c and suppose $\{x_n\}$ is a sequence in A with $\lim_{n\to\infty} x_n = c$ and $x_n \neq c$ for all $n \in \mathbb{N}$. Here, we prove that the sequence $\{f(x_n)\}$ converges to L.
Let $\epsilon > 0$ be given. Then by the definition of limit, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - c| < \delta$ and $x \in A$. Now we apply the definition of convergent sequence for the given $\delta > 0$ to obtain a natural number $K(\delta)$ such that if $n > K(\delta)$ then $|x_n - c| < \delta$. But for each such x_n, we have $|f(x_n) - L| < \epsilon$. Thus if $n > K(\delta)$, then $|f(x_n) - L| < \epsilon$. Therefore, the sequence $\{f(x_n)\}$ converges to L.
$(ii) \Rightarrow (i)$. If possible, suppose that (i) is not true. Then there exists an $\epsilon > 0$ such that for every $\delta > 0$ there exists a point $x \in A$, with $x \neq c$, for which $|f(x) - L| \geq \epsilon$ and $0 < |x - c| < \delta$. Let we take $\delta = \frac{1}{n}$. Then for each positive integer n, there exists $x_n \in A$, with $x_n \neq c$, such that $|f(x_n) - L| \geq \epsilon$ and $|x_n - c| < \delta = \frac{1}{n}$.
Thus, we get a sequence $\{x_n\}$ in $A \setminus \{c\}$ with $x_n \to c$ as $n \to \infty$, whereas $f(x_n) \not\to L$ as $n \to \infty$. Therefore we have shown that if (i) is not true, then (ii) is not true. Thus we conclude that (ii) implies (i). □

Divergence criteria:

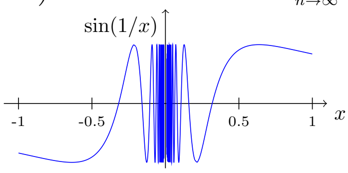
Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ and let c be a limit point of A.

(a) If $L \in \mathbb{R}$, then f does not have limit L at c if and only if there exists a sequence $\{x_n\}$ in A with $x_n \neq c$ for all $n \in \mathbb{N}$ such that the sequence $\{x_n\}$ converges to c but the sequence $\{f(x_n)\}$ does not converge to L.

(b) The function f does not have a limit at c if and only if there exists a sequence $\{x_n\}$ in A with $x_n \neq c$ for all $n \in \mathbb{N}$ such that the sequence $\{x_n\}$ converges to c but the sequence $\{f(x_n)\}$ does not converge in $\mathbb{R}$.

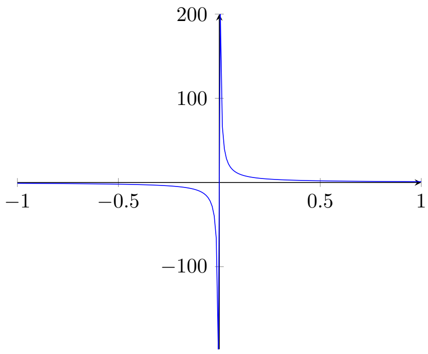
Example 1.9. $\lim_{x\to 0} \sin(\frac{1}{x})$ does not exist in $\mathbb{R}$.
Let $f(x) = \sin(\frac{1}{x})$ for $x \neq 0$. Now, consider two sequences $\{x_n\} := \{\frac{1}{n\pi}\}$ and $\{y_n\} := \left\{\frac{2}{(4n+1)\pi}\right\}$

for $n \in \mathbb{N}$. Then $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} y_n = 0$. Now, $f(x_n) = \sin(n\pi) = 0$ for all $n \in \mathbb{N}$, so that $\lim_{n\to\infty} f(x_n) = 0$. On the other hand $f(y_n) = \sin\left(\frac{(4n+1)\pi}{2}\right) = 1$ for all $n \in \mathbb{N}$, so that $\lim_{n\to\infty} f(y_n) = 1$.



So, we conclude that $\lim_{x\to 0} \sin(\frac{1}{x})$ does not exist in $\mathbb{R}$.

Example 1.10. $\lim_{x\to 0} \frac{1}{x}$ does not exist in $\mathbb{R}$.
Let $f(x) = \frac{1}{x}$ for $x \neq 0$. Now, consider the sequence $\{x_n\} := \{\frac{1}{n}\}$ for $n \in \mathbb{N}$. Then $\lim_{n\to\infty} x_n = 0$, but $f(x_n) = n$, which is not convergent in $\mathbb{R}$. So, form condition (b) $\lim_{x\to 0} \frac{1}{x}$ does not exist in $\mathbb{R}$.



Definition 1.11. Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ and let c be a limit point of A. We say that f is bounded on a neighborhood of c if there exists a δ-neighborhood $V_\delta(c) := \{x \in \mathbb{R} : \ |x - c| < \delta\}$ of c and a constant $M > 0$ such that $|f(x)| \le M$ for all $x \in A \cap V_\delta(c)$.

Theorem 1.12. *If $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ has a limit at $c \in \mathbb{R}$, then f is bounded on some neighborhood of c.*

Proof. Let $\lim_{x\to c} f(x) = L$. Then for $\epsilon = 1$, there exists $\delta > 0$ such that $|f(x) - L| < 1$ whenever $0 < |x - c| < \delta$ and $x \in A$. Hence, we have $|f(x)| - |L| \le |f(x) - L| < 1$. Therefore, if $x \in A \cap V_\delta(c)$, $x \neq c$, then $|f(x)| < |L| + 1$. If $c \notin A$, we take $M = |L| + 1$. If $c \in A$, we take $M = \max\{|f(c)|, |L| + 1\}$. It follows that if $x \in A \cap V_\delta(c)$, then $|f(x)| \le M$. This shows that f is bounded on the neighborhood of $V_\delta(c)$ of c. $\square$

Definition 1.13. Let $A \subseteq \mathbb{R}$ and let f and g be functions defined on A to $\mathbb{R}$. We define the sum $f + g$, the difference $f - g$, and the product fg on A to $\mathbb{R}$ to be the functions given by

$$(f + g)(x) := f(x) + g(x), \qquad (f - g)(x) := f(x) - g(x),$$
$$(fg)(x) := f(x)g(x),$$

for all $x \in A$. Further, if $b \in \mathbb{R}$, we define the multiple bf to be the function given by

$$(bf)(x) := bf(x) \text{ for all } x \in A.$$

Finally, if $h(x) \neq 0$ for all $x \in A$, we define the quotient $\frac{f}{h}$ to be the function given by

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$$\left(\frac{f}{h}\right)(x) := \frac{f(x)}{g(x)} \text{ for all } x \in A.$$

Theorem 1.14. *Let $A \subseteq \mathbb{R}$, let f and g be functions on A to $\mathbb{R}$ and let $c \in \mathbb{R}$ be a limit point of A. Further let $b \in \mathbb{R}$.*

(a) *If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, then:*

$$\lim_{x\to c}(f+g)(x) = L + M, \qquad \lim_{x\to c}(f-g)(x) = L - M,$$
$$\lim_{x\to c}(fg)(x) = LM, \qquad \lim_{x\to c}(bf)(x) = bL.$$

(b) *Let $h : A \to \mathbb{R}$ and let $h(x) \neq= 0$ for all $x \in A$. If $\lim_{x\to c} h(x) = H \neq 0$, then*

$$\lim_{x\to c}\left(\frac{f}{h}\right)(x) = \frac{L}{H}.$$

Theorem 1.15. ***(Squeeze Theorem)***
Let $A \subseteq \mathbb{R}$, let $f, g, h : A \to \mathbb{R}$ and let c be a limit point of A. Also let

$$f(x) \leq g(x) \leq h(x) \text{ for all } x \in A,\ x \neq c.$$

If $\lim_{x\to c} f(x) = L = \lim_{x\to c} h(x)$, then $\lim_{x\to c} g(x) = L$.

Proof. To prove this theorem we use the following results of sequence of real numbers:
Let $\{x_n\}$ and $\{y_n\}$ be two convergent sequence of real numbers. If $x_n \leq y_n$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n$.
Let $\{x_n\}$ be any sequence of real numbers such that $c \neq x_n \in A$ for all $n \in \mathbb{N}$. If the sequence $\{x_n\}$ converges to c, then by sequential criterion of limit and the above mentioned result we have

$$L = \lim_{n\to\infty} f(x_n) \leq \lim_{n\to\infty} g(x_n) \leq \lim_{n\to\infty} h(x_n) = L.$$

Therefore $\lim_{n\to\infty} g(x_n) = L = \lim_{x\to c} g(x)$. □

Theorem 1.16. *Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ and let c be a limit point of A. If*

$$\lim_{x\to c} f(x) > 0 \quad \left[\textit{respectively, } \lim_{x\to c} f(x) < 0\right],$$

then there exists a neighborhood $V_\delta(c)$ of c such that $f(x) > 0$ [respectively, $f(x) < 0$] for all $x \in A \cap V_\delta(c)$, $x \neq c$.

Proof. Let $\lim_{x\to c} f(x) = L$ and let $L > 0$. We take $\epsilon = \frac{L}{2} > 0$. Then there exists $\delta > 0$ such that $|f(x) - L| < \frac{L}{2}$, whenever $0 < |x - c| < \delta$. Therefore it follows that if $x \in A \cap V_\delta(c)$, $x \neq c$, then $f(x) > \frac{L}{2} > 0$.
If $L < 0$, a similar arguments applies. □

<u>One-sided Limits</u>

Definition 1.17. Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$.
(i) If $c \in \mathbb{R}$ is a limit point of the set $A \cap (c, \infty) = \{x \in A : x > c\}$, then we say that $L \in \mathbb{R}$ is right-hand limit of f at c and we write

$$\lim_{x\to c+} f(x) = L$$

if given any $\epsilon > 0$ there exists $\delta > 0$ (depending on ϵ and the point c) such that for all $x \in A$ with $0 < x - c < \delta$, then $|f(x) - L| < \epsilon$.

(ii) If $c \in \mathbb{R}$ is a limit point of the set $A \cap (-\infty, c) = \{x \in A : x < c\}$, then we say that $L \in \mathbb{R}$ is left-hand limit of f at c and we write

$$\lim_{x \to c-} f(x) = L$$

if given any $\epsilon > 0$ there exists $\delta > 0$ (depending on ϵ and the point c) such that for all $x \in A$ with $0 < c - x < \delta$, then $|f(x) - L| < \epsilon$.

Theorem 1.18. *Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ and let c be a limit point of $A \cap (c, \infty)$. Then the following statements are equivalent:*
(i) $\lim\limits_{x \to c+} f(x) = L$.
(ii) *For every sequence $\{x_n\}$ in A that converges to c such that $x_n > c$ for all $n \in \mathbb{N}$, the sequence $\{f(x_n)\}$ converges to L.*

Theorem 1.19. *Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ and let c be a limit point of $A \cap (-\infty, c)$. Then the following statements are equivalent:*
(i) $\lim\limits_{x \to c-} f(x) = L$.
(ii) *For every sequence $\{x_n\}$ in A that converges to c such that $x_n < c$ for all $n \in \mathbb{N}$, the sequence $\{f(x_n)\}$ converges to L.*

Theorem 1.20. *Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ and let c be a limit point of both the sets $A \cap (c, \infty)$ and $A \cap (-\infty, c)$. Then $\lim\limits_{x \to c} f(x) = L$ if and only if $\lim\limits_{x \to c-} f(x) = L = \lim\limits_{x \to c-} f(x)$.*

<u>**Infinite Limits**</u>

Definition 1.21. Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ and let c be a limit point of A.
(i) We say that f tends to ∞ as $x \to c$, and write

$$\lim_{x \to c} f(x) = \infty$$

if for every $\alpha \in \mathbb{R}$ there exists $\delta > 0$ (depending on α) such that for all $x \in A$ with $0 < |x - c| < \delta$, then $f(x) > \alpha$.
(ii) We say that f tends to $-\infty$ as $x \to c$, and write

$$\lim_{x \to c} f(x) = -\infty$$

if for every $\beta \in \mathbb{R}$ there exists $\delta > 0$ (depending on β) such that for all $x \in A$ with $0 < |x - c| < \delta$, then $f(x) < \beta$.

Theorem 1.22. *Let $A \subseteq \mathbb{R}$, let $f, g : A \to \mathbb{R}$ and let c be a limit point of A. Suppose that $f(x) \leq g(x)$ for all $x \in A,\ x \neq c$.*
(a) *If $\lim\limits_{x \to c} f(x) = \infty$, then $\lim\limits_{x \to c} g(x) = \infty$.*
(b) *If $\lim\limits_{x \to c} g(x) = -\infty$, then $\lim\limits_{x \to c} f(x) = -\infty$.*

Proof. (a) If $\lim\limits_{x \to c} f(x) = \infty$ and $\alpha \in \mathbb{R}$ is given, then there exists $\delta > 0$ (depending on α) such that for all $x \in A$ with $0 < |x - c| < \delta$, then $f(x) > \alpha$. Since $f(x) \leq g(x)$ for all $x \in A,\ x \neq c$, it follows that if $0 < |x - c| < \delta$ and $x \in A$, then $g(x) > \alpha$. Therefore $\lim\limits_{x \to c} g(x) = \infty$.
The proof of (b) is similar. □

Limits at Infinity

Definition 1.23. Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. Suppose that $(a, \infty) \subseteq A$ for some $a \in \mathbb{R}$. We say that $L \in \mathbb{R}$ is a limit of f as $x \to \infty$, and write

$$\lim_{x\to\infty} f(x) = L,$$

if given any $\epsilon > 0$ there exists $K > a$ (depending on ϵ) such that for any $x > K$, then $|f(x) - L| < \epsilon$.

Theorem 1.24. *Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ and let $(a, \infty) \subseteq A$ for some $a \in \mathbb{R}$. Then the following statements are equivalent:*
(i) $\lim_{x\to\infty} f(x) = L$.
(ii) For every sequence $\{x_n\}$ in $A \cap (a, \infty)$ such that $\lim_{n\to\infty}(x_n) = \infty$, the sequence $\{f(x_n)\}$ converges to L.

Definition 1.25. Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. Suppose that $(a, \infty) \subseteq A$ for some $a \in \mathbb{R}$. We say that f tends to ∞ [respectively $-\infty$] as $x \to \infty$, and write

$$\lim_{x\to\infty} f(x) = \infty, \ \left[\text{respectively } \lim_{x\to\infty} f(x) = -\infty\right]$$

if given any $\alpha \in \mathbb{R}$ there exists $K > a$ (depending on α) such that for any $x > K$, then $f(x) > \alpha$ [respectively $f(x) < \alpha$].

Theorem 1.26. *Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ and let $(a, \infty) \subseteq A$ for some $a \in \mathbb{R}$. Then the following statements are equivalent:*
(i) $\lim_{x\to\infty} f(x) = \infty$, *[respectively* $\lim_{x\to\infty} f(x) = -\infty$].
(ii) For every sequence $\{x_n\}$ in (a, ∞) such that $\lim_{n\to\infty}(x_n) = \infty$, then $\lim_{n\to\infty} f(x_n) = \infty$ [respectively, $\lim_{n\to\infty} f(x_n) = -\infty$].

Theorem 1.27. *Let $A \subseteq \mathbb{R}$, let $f, g : A \to \mathbb{R}$ and let $(a, \infty) \subseteq A$ for some $a \in \mathbb{R}$. Suppose further that $g(x) > 0$ for all $x > a$ and that for some $L \in \mathbb{R}$, $L \neq 0$, we have*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L.$$

(i) If $L > 0$, then $\lim_{x\to\infty} f(x) = \infty$ if and only if $\lim_{x\to\infty} g(x) = \infty$.
(ii) If $L < 0$, then $\lim_{x\to\infty} f(x) = -\infty$ if and only if $\lim_{x\to\infty} g(x) = -\infty$.

Proof. (i) Since $L > 0$, the hypothesis implies that there exists $a_1 > a$ such that

$$0 < \frac{1}{2}L \leq \frac{f(x)}{g(x)} < \frac{3}{2}L \text{ for} x > a_1.$$

Therefore we have $(\frac{1}{2}L)g(x) < f(x) < (\frac{3}{2}L)g(x)$ for all $x > a_1$, from which the conclusion follows readily.
The proof of (ii) is similar. □

2. **Continuity**

Definition 2.1. (Continuity of a function at a point)
Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ and let $c \in A$. Then f is said to be continuous at c if given any real number $\epsilon > 0$ there exists a real number $\delta > 0$ (depending on both ϵ and the point c) such that

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$$|f(x) - f(c)| < \epsilon \text{ whenever } |x - c| < \delta \text{ and } x \in A.$$

In terms of limit notion it means

$$\lim_{x \to c} f(x) = f(c).$$

Note.

- Let $f : [a, b] \to \mathbb{R}$ and let $c \in (a, b)$. Then f is said to be continuous at c if given any real number $\epsilon > 0$ there exists a real number $\delta > 0$ (depending on both ϵ and the point c) such that
$$|f(x) - f(c)| < \epsilon \text{ whenever } |x - c| < \delta \text{ and } x \in [a, b].$$
- Let $f : [a, b] \to \mathbb{R}$. Then f is said to be continuous at a if given any real number $\epsilon > 0$ there exists a real number $\delta > 0$ (depending on both ϵ and the point a) such that
$$|f(x) - f(a)| < \epsilon \text{ whenever } |x - a| < \delta \text{ and } x \in [a, b].$$
In terms of limit notion it means
$$\lim_{x \to a+} f(x) = f(a).$$
Similarly, f is said to be continuous at b if given any real number $\epsilon > 0$ there exists a real number $\delta > 0$ (depending on both ϵ and the point b) such that
$$|f(x) - f(b)| < \epsilon \text{ whenever } |x - b| < \delta \text{ and } x \in [a, b].$$
In terms of limit notion it means
$$\lim_{x \to b-} f(x) = f(b).$$

Definition 2.2. Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. Then f is said to be continuous on A if and only if f is continuous at each point of A.

Theorem 2.3. *Composition of two continuous functions is continuous.*

Proof. Let $f : A \to B$ and $g : B \to C$ be two continuous functions where $f(A) \subseteq B$. We want to show $g \circ f : A \to C$ is also continuous on A.
Let $x_0 \in A$ be an arbitrary point. Let $\epsilon > 0$ be any given real number. Then f is continuous at x_0 and g is continuous at $f(x_0) \in B$. So for $\epsilon > 0$ there exists a real number $\delta_1 > 0$ (depending on ϵ and $f(x_0)$) such that

$$|g(y) - g(f(x_0))| < \epsilon \text{ whenever } |y - f(x_0)| < \delta_1 \text{ and } y \in B. \ldots\ldots (1)$$

Let $\epsilon' = \delta_1$. Since f is continuous at x_0, then for $\epsilon' > 0$ there exists a real number $\delta > 0$ (depending on ϵ' and x_0) such that

$$|f(x) - f(x_0)| < \epsilon' \text{ whenever } |x - x_0| < \delta \text{ and } x \in A. \ldots\ldots\ldots (2)$$

Since $f(A) \subseteq B$, combining (1) and (2) it follows that

$$|g(f(x)) - g(f(x_0))| < \epsilon \text{ whenever } |x - x_0| < \delta \text{ and } x \in A.$$

$$\Rightarrow |(g \circ f)(x) - (g \circ f)(x_0)| < \epsilon \text{ whenever } |x - x_0| < \delta \text{ and } x \in A.$$

Therefore $g \circ f : A \to C$ is continuous at $x_0 \in A$. Since $x_0 \in A$ is arbitrary point, $g \circ f : A \to C$ is continuous on A. Thus composition of two continuous functions is continuous. □

Theorem 2.4. *Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$. Then f is continuous at a point $c \in A$ if and only if for every sequence $\{x_n\}$, $x_n \in A$ with $x_n \to c$ as $n \to \infty$, we get the sequence $\{f(x_n)\}$ converging to $f(c)$.*

Proof. Suppose f is continuous at $c \in A$. Let $\{x_n\}$ be a sequence converging to the point c, where $x_n \in A$, for each $n \in \mathbb{N}$. We show that $f(x_n) \to f(c)$ as $n \to \infty$.. Let $\epsilon > 0$ be an arbitrary real number. Since f is continuous at c, for $\epsilon > 0$ there exists a real number $\delta > 0$ (depending on $\epsilon > 0$ and c) such that $|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \delta$ and $x \in A$. . . . . (1)
Again $x_n \to c$, as $n \to \infty$. So for $\delta > 0$, there exists a positive integer n_0 such that

$$|x_n - c| < \delta \text{ for all } n \geq n_0 \ldots\ldots\ldots\ldots\ldots (2)$$

Combining (1) and (2) we get, $|f(x_n) - f(c)| < \epsilon$ for all $n \geq n_0$. Therefore $f(x_n) \to f(c)$ as $n \to \infty$.
Conversely, suppose for every sequence $\{x_n\}$ of real numbers, $x_n \in A$ with $x_n \to c$ as $n \to \infty$, we have $f(x_n) \to f(c)$ as $n \to \infty$. We want to show f is continuous at c. If possible, suppose that f is not continuous at c. Then there exists an $\epsilon > 0$ such that for every $\delta > 0$ there exists a point $x \in A$ for which $|f(x) - f(c)| \geq \epsilon$ and $|x - c| < \delta$. Let we take $\delta = \frac{1}{n}$. Then for each positive integer n there exists $x_n \in A$ such that $|f(x_n) - f(c)| \geq \epsilon$ and $|x - c| < \delta = \frac{1}{n}$.
Thus we get a sequence $\{x_n\}$ in A with $x_n \to c$ as $n \to \infty$, whereas $f(x_n) \not\to f(c)$ as $n \to \infty$. This is a contradiction to our assumption. So f must be continuous at c. □

Discontinuity criterion:

Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ and let $c \in A$. Then f is discontinuous at c if and only if there exists a sequence $\{x_n\}$ in A such that $\{x_n\}$ converges to c but the sequence $\{f(x_n)\}$ does not converge to $f(c)$.

Question 2.5. *Show that $f : \mathbb{R} \to \mathbb{R}$ defined by*

$$\begin{aligned} f(x) &= 1, \quad x \textit{ is rational} \\ &= 0, \quad x \textit{ is irrational} \end{aligned}$$

is discontinuous at every point of $\mathbb{R}$.

*Answer***.** Let c be a rational point. Then $f(c) = 1$. Since in any interval there are infinite number of rational as well as irrational numbers, for each positive integer n we can find an irrational number x_n such that $|x_n - c| < \frac{1}{n}$. Thus $x_n \to c$ as $n \to \infty$. But $f(x_n) = 0 \not\to f(c) = 1$, as $n \to \infty$. Therefore f is not continuous at c. As c is an arbitrary rational number, so f is not continuous at any rational number.
Again let d be any irrational number. Then $f(d) = 0$. Since in any interval there are infinite number of rational as well as irrational numbers, for each positive integer n we can find a rational number x_n such that $|x_n - d| < \frac{1}{n}$. Thus $x_n \to d$ as $n \to \infty$. But $f(x_n) = 1 \not\to f(d) = 0$, as $n \to \infty$. Therefore f is not continuous at d. As d is an arbitrary irrational number, so f is not continuous at any irrational number. Thus f is discontinuous everywhere.

Question 2.6. *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by*

$$\begin{aligned} f(x) &= x, \quad x \textit{ is irrational} \\ &= -x, \quad x \textit{ is rational} \end{aligned}$$

Show that f is continuous only at $x = 0$.

Answer. Let c be a rational number with $c \neq 0$. Since in any interval there are infinite number of rational as well as irrational numbers, for each positive integer n we can find an irrational number x_n such that $|x_n - c| < \frac{1}{n}$. Thus $x_n \to c$ as $n \to \infty$. But $f(x_n) = x_n$ for all n and so $f(x_n) \to c$ as $n \to \infty$. Since $f(c) = -c, f(x_n) \not\to f(c)$ as $n \to \infty$. Therefore f is discontinuous at c. Hence f is discontinuous at every non-zero rational points.

Let d be an irrational number. Then $f(d) = d$. Since in any interval there are infinite number of rational as well as irrational numbers, for each positive integer n we can find a rational number x_n such that $|x_n - d| < \frac{1}{n}$. Thus $x_n \to d$ as $n \to \infty$. But $f(x_n) = -x_n$ for all n and so $f(x_n) \to -d \neq f(d)$ as $n \to \infty$. Therefore f is discontinuous at d. Hence f is discontinuous at every irrational points.

We now show that f is continuous at $x = 0$. Let $\epsilon > 0$ be any real number. then there exists a real number $\delta = \epsilon > 0$ such that for all $x \in \mathbb{R}$,

$$|f(x) - f(0)| = |f(x)| = |x| < \epsilon \text{ whenever } |x - 0| < \delta.$$

Therefore f is continuous at $x = 0$.

Question 2.7. *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by*

$$\begin{aligned} f(x) &= 0,\ x \text{ is irrational} \\ &= \frac{1}{n},\ x \text{ is rational no. and } x = \frac{m}{n} \neq 0, \\ &\quad \text{where } n > 0 \text{ and } m, n \text{ are prime to each other} \\ &= 1,\ x = 0. \end{aligned}$$

Prove that f is continuous at every irrational point and that f has a simple discontinuity at every rational point.

Answer. Let c be an irrational number. Then $f(c) = 0$. Let $\epsilon > 0$ be any given real number. Then by Archimedean property there exists a natural number n_0 such that $\frac{1}{n_0} < \epsilon$. There are only a finite number of rationals with denominator less than n_0 in the interval $(c-1, c+1)$. Hence we can choose $\delta > 0$ so small that neighborhood $(c - \delta, c + \delta)$ contains no rational numbers with denominator less than n_0. So for all $x \in \mathbb{R}$ and $|x - c| < \delta$, we have

$$|f(x) - f(c)| = |f(x)| \leq \frac{1}{n_0} < \epsilon.$$

Therefore f is continuous at the point c. As c is an arbitrary irrational number, f is continuous at all irrational numbers.

Let d be a rational number. Then $f(d) > 0$. Since in any interval there are infinite number of rational as well as irrational numbers, for each positive integer n we can find an irrational number x_n such that $|x_n - d| < \frac{1}{n}$. Thus $x_n \to d$ as $n \to \infty$. But $f(x_n) = 0 \not\to f(d)$ as $n \to \infty$. Therefore f is discontinuous at d. Hence f is discontinuous at every rational points.

Remark 2.8. Suppose $f : A(\subseteq \mathbb{R}) \to \mathbb{R}$ be any function and $c \in A$ be a point, which is not a cluster (limit) point of A. So there exists a $\delta > 0$ such that $(c - \delta, c + \delta) \cap A = \{c\}$. Thus for given $\epsilon > 0$, we have $\delta > 0$ such that $|f(x) - f(c)| < \epsilon$, whenever $|x - c| < \delta$ and $x \in A$. So f is continuous at $c \in A$.

Discontinuities:

(A) Removal discontinuity:

A function $f : A(\subseteq \mathbb{R}) \to \mathbb{R}$ is said to be removal discontinuous at $x = c$ if $\lim_{x\to c} f(x)$ exists but f is not continuous at $x = c$. In this case, either $f(c)$ is not defined or $\lim_{x\to c} f(x) \neq f(c)$, when $f(c)$ is defined. Suppose $\lim_{x\to c} f(x) = L$. Then the function can be made continuous at $x = c$ either by assigning the value L to the function at $x = c$ or by changing the value of the function at $x = c$ to L.

Example 2.9.

- Let $f : (0, 1) \to \mathbb{R}$ be defined by $f(x) = x \sin\frac{1}{x}$. Then $\lim_{x\to 0} f(x) = 0$. So by assigning the value 0 to $f(0)$ we see that f is continuous at 0. Thus f has a removal discontinuity at $x = 0$.
- Let $f : [1, 3] \to \mathbb{R}$ be defined by

$$\begin{aligned} f(x) &= \frac{x^2-4}{x-2},\ x \neq 2 \\ &= 10,\ x = 2. \end{aligned}$$

  Here $\lim_{x\to 2} f(x) = 4 \neq 10 = f(2)$. By changing the value of the function at $x = 2$ from 10 to 4, i.e., $f(2) = 4$ we see that f is continuous at $x = 2$. Hence f is removal discontinuous at $x = 2$.

(B) Discontinuity of the first kind:

A function $f : A(\subseteq \mathbb{R}) \to \mathbb{R}$ is said to have a discontinuity of the first kind at $x = c$ if $\lim_{x\to c-} f(x)$ and $\lim_{x\to c+} f(x)$ both exist but are not equal.
f is said to have discontinuity of the first kind from the left at $x = c$ if $\lim_{x\to c-} f(x)$ exists but not equal to $f(c)$.
f is said to have discontinuity of the first kind from the right at $x = c$ if $\lim_{x\to c+} f(x)$ exists but not equal to $f(c)$.

Example 2.10.

- Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$\begin{aligned} f(x) &= 3,\ x > 2 \\ &= 2,\ x = 2 \\ &= 1,\ x < 2. \end{aligned}$$

  Then $\lim_{x\to 2+} f(x) = 3$ and $\lim_{x\to 2-} f(x) = 1$. So f has a discontinuity of the first kind at $x = 2$.
- $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = [x]\ \forall\ x \in \mathbb{R},$$

  where $[x]$ denotes the largest integer less than or equal to x. Then $f(x)$ is continuous at all non-integral points. At an integral point $x = n$, $\lim_{x\to n+} f(x) = n$ and $\lim_{x\to n-} f(x) = n - 1$. So $\lim_{x\to n-} f(x) \neq f(n) = n$. Therefore f has a discontinuity of the first kind from left at all integral points.
- Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$\begin{aligned} f(x) &= \frac{x-|x|}{x},\ x \neq 0 \\ &= 2,\ x = 0. \end{aligned}$$

Then $\lim\limits_{x\to+} f(x) = \lim\limits_{x\to 0+} \frac{x-x}{x} = 0$ and $\lim\limits_{x\to 0-} f(x) = \lim\limits_{x\to 0-} \frac{x+x}{x} = 2$. So $\lim\limits_{x\to 0+} f(x) \neq f(0) = 2$. Therefore f has a discontinuity of the first kind from right at $x = 0$.

(C) Discontinuity of the second kind:

A function $f : A(\subseteq \mathbb{R}) \to \mathbb{R}$ is said to have a discontinuity of the second kind at $x = c$ if neither $\lim\limits_{x\to c-} f(x)$ nor $\lim\limits_{x\to c+} f(x)$ exists.

f is said to have discontinuity of the second kind from the left at $x = c$ if $\lim\limits_{x\to c-} f(x)$ does not exist.

f is said to have discontinuity of the second kind from the right at $x = c$ if $\lim\limits_{x\to c+} f(x)$ does not exist.

Example 2.11.

- Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$\begin{aligned} f(x) &= \sin\frac{1}{x},\ x \neq 0 \\ &= 5,\ x = 0. \end{aligned}$$

  Then $\lim\limits_{x\to 0+} f(x)$ and $\lim\limits_{x\to 0-} f(x)$ both does not exist. Therefore, f has a discontinuity of the second kind at $x = 0$.
- Let $f : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ be defined by

$$\begin{aligned} f(x) &= \frac{1}{x}\sin\frac{1}{x},\ x \geq 0 \\ &= 0,\ x = 0. \end{aligned}$$

  Then $\lim\limits_{x\to 0+} f(x)$ does not exist and so f has a discontinuity of the second kind from right at $x = 0$.

(D) Infinite discontinuity:

Let $f : A(\subseteq \mathbb{R}) \to \mathbb{R}$. Then f is said to have an infinite discontinuity at $x = c$ if f is not bounded on any neighborhood of c.

Example 2.12.

- Let $f : \mathbb{R}^+ \to \mathbb{R}$ be defined by

$$f(x) = \log x,\ x > 0.$$

  Then $\lim\limits_{x\to 0+} f(x) = -\infty$ and so f has infinite discontinuity at $x = 0$.
- Let $f : \mathbb{R}^+ \to \mathbb{R}$ be defined by

$$f(x) = \frac{1}{x},\ x > 0.$$

  Then $\lim\limits_{x\to 0+} f(x) = \infty$ and so f has infinite discontinuity at $x = 0$.
- Let $f : \mathbb{R}^+ \to \mathbb{R}$ be defined by

$$f(x) = \frac{1}{x}\left|\sin\frac{1}{x}\right|,\ x > 0.$$

  Then $\lim\limits_{x\to 0+} f(x)$ does not exist but f is not bounded on any neighborhood of $x = 0$.

(E) Jump discontinuity:

Let $f : A(\subseteq \mathbb{R}) \to \mathbb{R}$. Then f is said to have a jump discontinuity at $x = c$ if $\lim_{x\to c-} f(x)$ and $\lim_{x\to c+} f(x)$ exist and $\lim_{x\to c-} f(x) \neq \lim_{x\to c+} f(x)$. The jump of f at c is defined as

$$J_f(c) = f(c+) - f(c-).$$

Clearly f has a removal discontinuity if $J_f(c) = 0$ and jump discontinuity if $J_f(c) \neq 0$.
Let $f : [a, b] \to \mathbb{R}$. If $\lim_{x\to a+} f(x)$ exists but not equal to $f(a)$ then we say that f has a right hand jump at $x = a$, the jump of f being $f(a+) - f(a)$.
Let $f : [a, b] \to \mathbb{R}$. If $\lim_{x\to b-} f(x)$ exists but not equal to $f(b)$ then we say that f has a left hand jump at $x = b$, the jump of f being $f(b) - f(b-)$.

Example 2.13. Let $f : [1, 2] \to \mathbb{R}$ be defined by

$$\begin{aligned} f(x) &= 4,\ x \in [1, 1.5) \\ &= 3,\ x = 1.5 \\ &= 2,\ x \in (1.5, 2]. \end{aligned}$$

Then f has a jump discontinuity at $x = 1.5$.

Question 2.14. *Let $a < b < c$. Suppose f is continuous on $[a, b]$ and g is continuous on $[b, c]$ and $f(b) = g(b)$. Define h on $[a, c]$ by $h(x) = f(x)$ if $x \in [a, b]$ and $h(x) = g(x)$ if $x \in [b, c]$. Prove that h continuous on $[a, c]$.*

Question 2.15. *Determine the points of continuity of the following functions:*
(i) $f(x) = x[x];\ x \in \mathbb{R}$, *(ii)* $g(x) = [\sin x];\ x \in \mathbb{R}$,
(iii) $h(x) = \left[\frac{1}{x}\right];\ x \neq 0, x \in \mathbb{R}$, *(iv)* $k(x) = x - [x];\ x \in \mathbb{R}$.

Theorem 2.16. *Let $f : K \to \mathbb{R}$ be a continuous function and $K \subseteq \mathbb{R}$ be compact set. Then $f(K)$ is compact set in $\mathbb{R}$.*

Proof. Let $\{y_n\}$ be a sequence in $f(K)$. Then for each positive integer n there exists a real number $x_n \in K$ such that $f(x_n) = y_n$. As K is compact, $\{x_n\} \subseteq K$ has a convergent subsequence $\{x_{n_k}\}$ converging to some point, say $x \in K$. As f is continuous, $x_{n_k} \to x$ implies $f(x_{n_k}) \to f(x) \in f(K)$, as $k \to \infty$. Therefore, y_{n_k} converges to $f(x) \in f(K)$. Since $\{y_n\}$ is arbitrary sequence, $f(K)$ is compact. □

Theorem 2.17. *Let $f : \mathbb{R} \to \mathbb{R}$. Then f is continuous if and only if $f^{-1}(G)$ is open set in $\mathbb{R}$ for every open set G in $\mathbb{R}$.*

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. We want to show $f^{-1}(G)$ is open set in $\mathbb{R}$, i.e., every point in $f^{-1}(G)$ is an interior point. Let $x_0 \in f^{-1}(G) = \{x \in \mathbb{R} : f(x) \in G\}$. Then $f(x_0) \in G$. As G is open, there exists a real number $\epsilon > 0$ such that $(f(x_0) - \epsilon, f(x_0) + \epsilon) \subseteq G$. Since f is continuous, for $\epsilon > 0$ there exists $\delta > 0$ such that $f(x) \in (f(x_0) - \epsilon, f(x_0) + \epsilon)$ whenever $x \in (x_0 - \delta, x_0 + \delta)$. Hence $f(x_0 - \delta, x_0 + \delta) \subseteq (f(x_0) - \epsilon, f(x_0) + \epsilon) \subseteq G$, i.e., $(x_0 - \delta, x_0 + \delta) \subseteq f^{-1}(f(x_0) - \epsilon, f(x_0) + \epsilon) \subseteq f^{-1}(G)$. Therefore, x_0 is an interior point of $f^{-1}(G)$. So, $f^{-1}(G)$ is open set in $\mathbb{R}$, as x_0 ia an arbitrary point.

Conversely, let $f^{-1}(G)$ be open set in $\mathbb{R}$ for every open set G in $\mathbb{R}$. We want to show f is continuous. Let $x_0 \in \mathbb{R}$ be arbitrary and let $\epsilon > 0$ be given any real number. Then $(f(x_0)-\epsilon, f(x_0)+\epsilon)$ be a open set containing $f(x_0)$. Then $f^{-1}(f(x_0)-\epsilon, f(x_0)+\epsilon)$ is open in $\mathbb{R}$ and $x_0 \in f^{-1}(f(x_0)-\epsilon, f(x_0)+\epsilon)$. Therefore, there exists a real number $\delta > 0$ such that $(x_0-\delta, x_0+\delta) \subseteq f^{-1}(f(x_0)-\epsilon, f(x_0)+\epsilon)$, i.e., $f(x_0-\delta, x_0+\delta) \subseteq (f(x_0)-\epsilon, f(x_0)+\epsilon)$. Hence f is continuous at x_0. So f is continuous on $\mathbb{R}$, as x_0 is an arbitrary point. □

Theorem 2.18. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f is bounded.*

Proof. We want to show that there exists a constant $k > 0$ such that $|f(x)| \leq k$ for all $x \in [a, b]$. If possible let f be not bounded on $[a, b]$. Then for each positive integer n we can find a point $x_n \in [a, b]$ such that $f(x_n) > n$. Since $[a, b]$ is bounded, the sequence $\{x_n\}$ is bounded. By Bolzano-Weierstrass Theorem $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ converging to some point, say x. Since the set $[a, b]$ is closed and all the elements of the sequence $\{x_{n_k}\}$ belong to $[a, b]$, we get $x \in [a, b]$. Now f is continuous at x and $\{x_{n_k}\}$ converges to x. So, we must have $\{f(x_{n_k})\}$ converges to $f(x)$. Thus $\{f(x_{n_k})\}$ being convergent is bounded. But $|f(x_{n_k})| > n_k \geq k$ for all $k = 1, 2, \ldots$, which contradicts the fact that $\{f(x_{n_k})\}$ is bounded. Hence the supposition that f is not bounded is wrong. So f is bounded on $[a, b]$. □

Remark 2.19. The Theorem 2.18 does not hold if the closed interval $[a, b]$ is replaced by a non-closed interval.

Example 2.20. Let $f : (0, 1) \to \mathbb{R}$ be defined by $f(x) = \frac{1}{x}$. Then clearly f is continous on $(0, 1)$ but f is not bounded on $(0, 1)$.

Theorem 2.21. *A function f, continuous on $[a, b]$, attains its bounds at least once in $[a, b]$.*

Proof. Since f is continuous on $[a, b]$, it is bounded on $[a, b]$, i.e., the set $\{f(x) : a \leq x \leq b\}$ is bounded. So the set $\{f(x) : a \leq x \leq b\}$ has a supremum and a infimum which we denote by $\sup f$ and $\inf f$ respectively.
We want to show that there exist points c, d in $[a, b]$ such that $f(c) = \sup f = \sup\{f(x) : a \leq x \leq b\}$ and $f(d) = \inf f = \inf\{f(x) : a \leq x \leq b\}$. We first prove the result for $\sup f$. let $M = \sup f$. If possible let there exists no $x \in [a, b]$ such that $f(x) = M$.
Then consider the function $g : [a, b] \to \mathbb{R}$ defined by $g(x) = M - f(x)$ for all $x \in [a, b]$. Clearly g is continuous on $[a, b]$ and $g(x) > 0$ for all $x \in [a, b]$. So $\frac{1}{g}$ is also continuous on $[a, b]$ and hence bounded on $[a, b]$. Thus there exists a constant $k > 0$ such that

$$\begin{aligned}
\frac{1}{g}(x) &< k \text{ for all } x \in [a, b] \\
\Rightarrow g(x) &> \frac{1}{k} \text{ for all } x \in [a, b] \\
\Rightarrow M - f(x) &> \frac{1}{k} \text{ for all } x \in [a, b] \\
\Rightarrow f(x) &< M - \frac{1}{k} \text{ for all } x \in [a, b].
\end{aligned}$$

Thus we get a contradiction to the fact that $M = \sup f$. Therefore, there exists at least one $x \in [a, b]$ for which $f(x) = M$. Thus f attains its supremum at least once in $[a, b]$.

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The result for infimum follows as a consequence because the infimum of f is the supremum of $-f$. This completes the proof of the theorem. □

Remark 2.22. If a function f is continuous on $[a, b]$, then there exist points $c, d \in [a, b]$ such that $m = f(c) \leq f(x) \leq f(d) = M$ for all $x \in [a, b]$. So $f([a, b]) \subseteq [f(c), f(d)]$.

Remark 2.23. The Theorem 2.21 does not hold if the closed interval $[a, b]$ is replaced by a non-closed interval.

Example 2.24. Let $f : (0, 1) \to \mathbb{R}$ be defined by $f(x) = x^2$. Then f is bounded on $(0, 1)$ as $0 < f(x) < 1$ for all $x \in (0, 1)$. Also $\sup f = 1$ and $\inf f = 0$. Hence f does not attain its bound on $(0, 1)$.

Theorem 2.25. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$. If $f(a)f(b) < 0$, then there exists $c \in (a, b)$ such that $f(c) = 0$.*

Proof. We assume that $f(a) < 0 < f(b)$. We will generate a sequence of intervals by successive bisections. Let $I_1 = [a_1, b_1]$, where $a_1 = a,\ b_1 = b$. Let p_1 be the midpoint of a_1 and b_1, i.e., $p_1 = \frac{1}{2}(a_1 + b_1)$. If $f(p_1) = 0$, we take $c = p_1$ and we are done. If $p_1 \neq 0$, then either $f(p_1) > 0$ or $f(p_1) < 0$. If $f(p_1) > 0$, then we set $a_2 = a_1,\ b_2 = p_1$, while $f(p_1) < 0$, then we set $a_2 = p_1,\ b_2 = b_1$. In either case, we let $I_2 = [a_2, b_2]$. Then we have $I_2 \subsetneq I_1$ and $f(a_2) < 0,\ f(b_2) > 0$.
We continue the bisection process. Suppose that the intervals $I_1, I_2, \ldots, I_k$ have been obtained by successive bisection in the same manner. Then we have $f(a_k) < 0$ and $f(b_k) > 0$ and we set $p_k = \frac{1}{2}(a_k + b_k)$. If $f(p_k) = 0$, we take $c = p_k$ and we are done. If $p_k > 0$, then we set $a_{k+1} = a_k,\ b_{k+1} = p_k$, while $f(p_k) < 0$, then we set $a_{k+1} = p_k,\ b_{k+1} = b_k$. In either case, we let $I_{k+1} = [a_{k+1}, b_{k+1}]$. Then we have $I_{k+1} \subsetneq I_k$ and $f(a_{k+1}) < 0,\ f(b_{k+1}) > 0$.
If the process terminates by locating a point p_n such that $f(p_n) = 0$, then we are done. If the process does not terminate, then we obtain a nested sequence of closed bounded intervals $\{I_n\} = \{[a_n, b_n]\}$ such that for every $n \in \mathbb{N}$ we have $f(a_n) < 0$ and $f(b_n) > 0$. Furthermore, since the intervals are obtained by repeated bisection, the length of I_n is equal to $(b_n - a_n) = \frac{(b-a)}{2^{n-1}}$. It follows from the Nested Intervals Property that there exists a point c that belongs to I_n for every $n \in \mathbb{N}$. Since $a_n \leq c \leq b_n$ for all $n \in \mathbb{N}$, we have $0 \leq c - a_n \leq b_n - a_n = \frac{(b-a)}{2^{n-1}}$ and $0 \leq b_n - c \leq b_n - a_n = \frac{(b-a)}{2^{n-1}}$. Hence, it follows that $\lim_{n\to\infty}(a_n) = c = \lim_{n\to\infty}(b_n)$. Since f is continuous at c, we have

$$\lim_{n\to\infty}(f(a_n)) = f(c) = \lim_{n\to\infty}(f(b_n)).$$

The fact that $f(a_n) < 0$ for all $n \in \mathbb{N}$ implies that $f(c) = \lim_{n\to\infty}(f(a_n)) \leq 0$. Also, the fact that $f(b_n) > 0$ for all $n \in \mathbb{N}$ implies that $f(c) = \lim_{n\to\infty}(f(b_n)) \geq 0$. Thus, we conclude that $f(c) = 0$. □

Theorem 2.26. ***(Bolzano's Intermediate Value Theorem)***
Let I be an interval and let $f : I \to \mathbb{R}$ be continuous on I. If $a, b \in I$ and if $k \in \mathbb{R}$ satisfies $f(a) < k < f(b)$, then there exists a point $c \in I$ between a and b such that $f(c) = k$.

Proof. Suppose that $a < b$ and let $g(x) := f(x) - k$. Then $g(a) < 0 < g(b)$. By the Theorem 2.25 there exists a point c with $a < c < b$ such that $0 = g(c) = f(c) - k$. Therefore $f(c) = k$.
If $b < a$, let $h(x) := k - f(x)$ so that $h(b) < 0 < h(a)$. Therefore there exists a point c with $a < c < b$ such that $0 = h(c) = k - f(c)$. Therefore $f(c) = k$.

□

Monotone and Inverse functions:
Monotone functions are not always continuous.

Example 2.27. Let $f : [0, 3] \to \mathbb{R}$ be defined by

$$\begin{aligned} f(x) &= 2, \ x \in [0, 2] \\ &= 3, \ x \in (2, 3]. \end{aligned}$$

Then clearly f is increasing on $[0, 3]$ but f is not continuous at $x = 2$, as $f(2-) = 2$ and $f(2+) = 3$.

The next theorem shows that monotone functions defined on interval always has one sided limits in $\mathbb{R}$ at every point that is not an end point of its domain.

Theorem 2.28. *Let $f : [a, b] \to \mathbb{R}$ be an increasing function and $c \in (a, b)$. Then*

$$(i) \ f(c-) = \sup\{f(x) : a \le x < c\}$$

$$(ii) \ f(c+) = \inf\{f(x) : c < x \le b\}.$$

Proof. Clearly for $a \le x < c$, we have $f(x) \le f(c)$. So the set $\{f(x) : a \le x < c\}$ is bounded above by $f(c)$. Hence $\sup\{f(x) : a \le x < c\}$ exists. Let $M = \sup\{f(x) : a \le x < c\}$.
Let $\epsilon > 0$ be any given real number. Then there exists an element x_ϵ, $a \le x_\epsilon < c$ such that $M - \epsilon < f(x_\epsilon) \le M$. Let $\delta = c - x_\epsilon$. Then $\delta > 0$. Now for all $x \in (c - \delta, c)$ and $x \in [a, b]$ we have $x_\epsilon < x < c$ and $a \le x \le b$. Therefore, we have $M - \epsilon < f(x_\epsilon) \le f(x) \le M < M + \epsilon \Rightarrow |f(x) - M| < \epsilon$. Thus $|f(x) - M| < \epsilon$ whenever $x \in (c - \delta, c) \cap [a, b]$. Therefore,

$$\lim_{x \to c-} f(x) = M = \sup\{f(x) : a \le x < c\}.$$

Similarly, one can show that $\lim_{x \to c+} f(x) = \inf\{f(x) : c < x \le b\}$. □

Note. The result holds for any non-closed interval also.

Theorem 2.29. *Let $f : [a, b] \to \mathbb{R}$ be a decreasing function and $c \in (a, b)$. Then*

$$(i) \ f(c-) = \lim_{x \to c-} f(x) = \inf\{f(x) : a \le x < c\}$$

$$(ii) \ f(c+) = \lim_{x \to c+} f(x) = \sup\{f(x) : c < x \le b\}.$$

Remark 2.30. From the last two theorems it follows that monotonic functions have no discontinuities of the second kind.

Theorem 2.31. *Let f be monotonic on (a, b). Then the set of points of (a, b) at which f is discontinuous is countable.*

Proof. Suppose that f is increasing on (a, b). Let D be the set of points at which f is discontinuous. Then with every point x of D we associate a rational number $r(x)$ such that $f(x-) < r(x) < f(x+)$. Now for any two points $x_1, x_2 \in D$ with $x_1 < x_2$, we have $f(x_1+) \le f(x_2-)$. So $r(x_1) \ne r(x_2)$ if $x_1 \ne x_2$. Thus there exists a one-one correspondence between the set D and a subset of rational numbers. As the set of rational numbers is countable, D is also countable. □

Theorem 2.32. ***(Continuous Inverse Theorem)***
Let $I \subset \mathbb{R}$ be a closed and bounded interval and let $f : I \to \mathbb{R}$ be strictly monotone and continuous on I. Then the function g, inverse of f is strictly monotone and continuous on $J = f(I)$.

Proof. Suppose f is strictly increasing on I and also suppose $I = [a, b]$. Since f is continuous on $[a, b]$, there exists points $c, d \in [a, b]$ such that $f(c) = \inf\{f(x) : a \le x \le b\}$, $f(d) = \sup\{f(x) : a \le x \le b\}$ and $f([a, b]) = [f(c), f(d)]$. Thus range of f is also a closed and bounded interval. We now show that $f^{-1} : [f(c), f(d)] \to [a, b]$ exists.
Let $x_1, x_2 \in [a, b]$ and $x_1 \neq x_2$. If $x_1 < x_2$, then $f(x_1) < f(x_2)$. If $x_2 < x_1$, then $f(x_2) < f(x_1)$. Thus in both cases $f(x_1) \neq f(x_2)$. So, f is injective. Thus $f : [a, b] \to [f(c), f(d)]$ is a bijective function. Hence f^{-1} exists. Let $f^{-1} = g$.
We now show that g is strictly increasing. For this let $y_1, y_2 \in [f(c), f(d)]$ and $y_1 < y_2$. Then there exist $x_1, x_2 \in [a, b]$ such that $f(x_1) = y_1, f(x_2) = y_2$. As f is strictly increasing, we must have $x_1 < x_2$, otherwise if $x_1 \ge x_2$ then $f(x_1) \ge f(x_2) \Rightarrow y_1 \ge y_2$, which contradicts the fact that $y_1 < y_2$. Therefore $y_1 < y_2 \Rightarrow g(y_1) = x_1 < x_2 = g(y_2)$. So, g is strictly increasing.
Finally we show that g is continuous on $[f(c), f(d)]$. If possible suppose that g is not continuous at some point $y_0 \in [f(c), f(d)]$. Then $\lim\limits_{y \to y_0-} g(y) < \lim\limits_{y \to y_0+} g(y)$, as g is strictly increasing. Choose a number $x \neq g(y_0)$ such that $g(y_0-) < x < g(y_0+)$. Then $x \neq g(y)$ for any $y \in [f(c), f(d)]$. Hence $x \notin [a, b]$. This contradicts the fact that $g([f(c), f(d)]) = [a, b]$ is an interval. Therefore g is continuous. □

Theorem 2.33. *Let $I \subset \mathbb{R}$ be a closed and bounded interval and let $f : I \to \mathbb{R}$ be an injective continuous function on I. Then f is strictly monotone on I.*

Proof. Suppose $I = [a, b]$. Since f is injective on $[a, b]$, we must have either $f(a) < f(b)$ or $f(a) > f(b)$. Suppose that $f(a) < f(b)$. We then show that f is strictly increasing on $[a, b]$. Let $x \in (a, b)$. If $f(x) < f(a) < f(b)$, then by applying Bolzano's intermediate value theorem to the function f on $[x, b]$, we get a point $a' \in (x, b)$ such that $f(a) = f(a')$. This contradicts the fact that f is injective. So the relation $f(x) < f(a)$ can't hold.
Simliarly if $f(a) < f(b) < f(x)$, then by applying Bolzano's intermediate value theorem to the function f on $[a, x]$, we get a point $b' \in (a, x)$ such that $f(b) = f(b')$. This contradicts the fact that f is injective. So the relation $f(b) < f(x)$ can't hold.
Thus $f(a) < f(x) < f(b)$ for all $x \in (a, b)$. Now let $y \in (a, b)$ and $x < y$. Then $f(a) < f(y) < f(b)$. If $f(a) < f(y) < f(x)$, then by preceding arguments there exists $y' \in (a, x)$ such that $f(y) = f(y')$, which contradicts the fact that f is injective. Therefore $f(y) > f(x)$. Then for $x, y \in (a, b)$, with $x < y$, we get $f(x) < f(y)$. Hence f is strictly increasing on $[a, b]$.
Similarly if $f(a) > f(b)$, then we can show that f is strictly decreasing on $[a, b]$. □

3. Uniform Continuity

Definition 3.1. Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. We say that f is uniformly continuous on A if for each $\epsilon > 0$ there exists $\delta > 0$ (depending on ϵ) such that if $x, y \in A$ are any numbers satisfying $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

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Theorem 3.2. *Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. Then the following statements are equivalent:*
(i) f is uniformly continuous on A.
(ii) Every pair of sequence $\{x_n\}$ and $\{y_n\}$ in A with $\lim_{n\to\infty}(x_n - y_n) = 0$ implies $\lim_{n\to\infty}(f(x_n) - f(y_n)) = 0$.

Proof. $(i) \Rightarrow (ii)$. Assume f is uniform continuous on A and suppose $\{x_n\}$ and $\{y_n\}$ are sequences in A with $\lim_{n\to\infty}(x_n - y_n) = 0$. Here, we prove that the sequence $\{f(x_n) - f(y_n)\}$ converges to 0.
Let $\epsilon > 0$ be given. Then by the definition of uniform convergence, there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$ and $x, y \in A$. Now we apply the definition of convergent sequence for the given $\delta > 0$ to obtain a natural number $K(\delta)$ such that if $n > K(\delta)$ then $|(x_n - y_n) - 0| < \delta$. But for each such $(x_n - y_n)$, we have $|f(x_n) - f(y_n)| < \epsilon$. Thus if $n > K(\delta)$, then $|f(x_n) - f(y_n)| < \epsilon$. Therefore, the sequence $\{f(x_n) - f(y_n)\}$ converges to 0.
$(ii) \Rightarrow (i)$. If possible, suppose that (i) is not true. Then there exists an $\epsilon > 0$ such that for every $\delta > 0$ there exists points $x_\delta, y_\delta \in A$, such that $|x_\delta - y_\delta| < \delta$ and $|f(x_\delta) - f(y_\delta)| \geq \epsilon$. Let we take $\delta = \frac{1}{n}$. Then for each positive integer n, there are points $x_n, y_n \in A$, such that $|f(x_n) - f(y_n)| \geq \epsilon$ and $|x_n - y_n| < \delta = \frac{1}{n}$.
Thus, we get two sequences $\{x_n\}$ and $\{y_n\}$ in A with $(x_n - y_n) \to 0$ as $n \to \infty$, whereas $(f(x_n) - f(y_n) \not\to 0$ as $n \to \infty$. Therefore we have shown that if (i) is not true, then (ii) is not true. Thus we conclude that (ii) implies (i). □

Example 3.3.

- Let $f : (0, 1) \to \mathbb{R}$ be defined by $f(x) = \frac{1}{x}$. consider two sequences $\{x_n\} := \left\{\frac{1}{n+1}\right\}$ and $\{y_n\} := \left\{\frac{1}{n+2}\right\}$. Then $\lim_{n\to\infty}(x_n - y_n) = 0$, but $\lim_{n\to\infty}(f(x_n) - f(y_n)) = 1$. Therefore by Theorem 3.2, we conclude that f is not uniformly continuous on $(0, 1)$.

- Let $g : (0, 1) \to \mathbb{R}$ be defined by $g(x) = \sin(\frac{1}{x})$. consider two sequences $\{x_n\} := \left\{\frac{2}{(4n+1)\pi}\right\}$ and $\{y_n\} := \left\{\frac{1}{n\pi}\right\}$. Then $\lim_{n\to\infty}(x_n - y_n) = 0$, but $\lim_{n\to\infty}(f(x_n) - f(y_n)) = 1$. Therefore by Theorem 3.2, we conclude that g is not uniformly continuous on $(0, 1)$.

- Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $h(x) = x^2$. consider two sequences $\{x_n\} := \left\{n + \frac{1}{n}\right\}$ and $\{y_n\} := \{n\}$. Then $\lim_{n\to\infty}(x_n - y_n) = 0$, but $\lim_{n\to\infty}(f(x_n) - f(y_n)) = 2$. Therefore by Theorem 3.2, we conclude that h is not uniformly continuous on $\mathbb{R}$.

Theorem 3.4. *Let $I \subset \mathbb{R}$ be a closed and bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. Then f is uniformly continuous on I.*

Proof. If f is not uniformly continuous on I, then there exists an $\epsilon > 0$ and two sequences $\{x_n\}$ and $\{y_n\}$ in I such that $|x_n - y_n| < \frac{1}{n}$ and $|f(x_n) - f(y_n)| \geq \epsilon$ for all $n \in \mathbb{N}$. Since I is bounded, the sequence $\{x_n\}$ is bounded. Therefore, by the Bolzano-Weierstrass Theorem there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges to a point x. Since I is closed, the limit x belongs to I. Hence the corresponding subsequence $\{y_{n_k}\}$ also converges to x, since

$$|y_{n_k} - x| \leq |y_{n_k} - x_{n_k}| + |x_{n_k} - x|.$$

Now, if f is continuous at the point x, then both the sequences $\{f(x_{n_k})\}$ and $\{f(y_{n_k})\}$ must converge to $f(x)$. But this is not possible, since

$$|f(x_n) - f(y_n)| \geq \epsilon,$$

for all $n \in \mathbb{N}$. Thus the hypothesis that f is not uniformly continuous on the closed bounded interval I implies that f is not continuous at some point $x \in I$. Consequently, if f is continuous at every point of I, then f uniformly continuous on I. □

Definition 3.5. (Lipschitz Functions)
Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. If there exists a constant $K > 0$ such that

$$|f(x) - f(y)| \leq K|x - y|$$

for all $x, y \in A$, then f is said to be a Lipschitz function (or to satisfy a Lipschitz condition) on A.

Theorem 3.6. *Ia $f : A \to \mathbb{R}$ is a Lipschitz function, then f is uniformly continuous on A.*

Proof. Since f is a Lipschitz function, there exists $K > 0$ such that

$$|f(x) - f(y)| \leq K|x - y|$$

for all $x, y \in A$. Now, for given any $\epsilon > 0$, we can take $\delta = \frac{\epsilon}{K}$. If $x, y \in A$ satisfy $|x - y| < \delta$, then $|f(x) - f(y)| < K\frac{\epsilon}{K} = \epsilon$. Therefore f is uniformly continuous on A. □

Example 3.7. If $f(x) = x^2$ on $A = [0, b]$, where $b > 0$, then

$$|f(x) - f(y)| = |x^2 - y^2| = |x + y||x - y| \leq 2b|x - y|$$

for all $x, y \in A$. Thus f satisfies Lipschitz condition with $K = 2b$. Therefore f is uniformly continuous on $[0, b]$ by Theorem 3.6.

Remark 3.8. The converse of the Theorem 3.6 is not true in general.

Example 3.9. Let $f(x) = \sqrt{x}$, on the bounded and closed interval $I = [0, 2]$. Then by Theorem 3.4, f is uniformly continuous on $[0, 2]$, but f does not satisfy Lipschitz condition on $[0, 2]$, as

$$|f(x) - f(y)| = |\sqrt{x} - \sqrt{y}| = \frac{|x - y|}{\sqrt{x} + \sqrt{y}} \to \infty, \text{ as } x, y \to 0.$$

Definition 3.10. (Cauchy Sequence)
A sequence $\{x_n\} \subseteq \mathbb{R}$ is said to be Cauchy sequence if for given $\epsilon > 0$ there exists $K \in \mathbb{N}$ (depending on ϵ) such that $|x_m - x_n| < \epsilon$, whenever $m, n \geq K$.

Theorem 3.11. *If $f : A \to \mathbb{R}$ is uniformly continuous on a subset A of $\mathbb{R}$ and $\{x_n\}$ is a Cauchy sequence in A, then $\{f(x_n)\}$ is a Cauchy sequence in $\mathbb{R}$.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in A and let $\epsilon > 0$ be given. First choose $\delta > 0$ such that if $x, y \in A$ satisfy $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$. Since $\{x_n\}$ is a Cauchy sequence, there exists $K \in \mathbb{N}$ (depending on δ) such that $|x_m - x_n| < \delta$, whenever $m, n \geq K$. By the choice of δ, this implies that for $m, n \geq K$, we have $|f(x_m) - f(x_n)| < \epsilon$. Therefore $\{f(x_n)\}$ is a Cauchy sequence in $\mathbb{R}$. □

Example 3.12. Let $f : (0,1) \to \mathbb{R}$ be defined by $f(x) = \frac{1}{x}$. Consider the sequence $\{x_n\} = \{\frac{1}{n+1}\}$ in $(0,1)$. The sequence $\{x_n\} = \{\frac{1}{n+1}\}$ Cauchy in $(0,1)$, but the sequence $\{f(x_n)\} = \{n+1\}$ is not Cauchy in $\mathbb{R}$. Hence, by Theorem 3.11, we conclude that f is not uniformly continuous on $(0,1)$.

Theorem 3.13. *A function $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous on the interval (a,b) if and only if it can be defined at the end points a and b such that the extended function is continuous on $[a,b]$.*

Proof. " If " part easily follows from Theorem 3.4.
Here, we prove the " only if " part of the theorem. Suppose f is uniformly continuous on (a,b). We shall show how to extend f to a continuously; the argument for b is similar. This is done by showing that $\lim_{x\to a} f(x) = L$ (say) exists and this is accomplished by using the sequential criterion for limits. If $\{x_n\}$ is a sequence in (a,b) with $\lim_{n\to\infty} x_n = a$, then it is a Cauchy sequence. By the Theorem 3.11, the sequence $\{f(x_n)\}$ is a Cauchy sequence. We know that a sequence in real numbers is convergent if and only if it is a Cauchy sequence. Therefore the limit $\lim_{n\to\infty} f(x_n) = L$ (say) exists. If $\{y_n\}$ is any other sequence in (a,b) that converges to a, then $\lim_{n\to\infty}(y_n - x_n) = a - a = 0$, so by the uniform continuity of we have

$$\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty}(f(y_n) - f(x_n)) + \lim_{n\to\infty} f(x_n) = 0 + L = L.$$

Since we get the same value L every sequence converging to a, we infer from the sequential criterion for limits that f has limit L at a. Now, if we define $f(a) := L$, then f is continuous at a. The same argument applies to b, so we conclude that f has a continuous extension to the interval $[a,b]$.
This completes the proof of the theorem. □

Example 3.14.

- Since the limit of $f(x) := \sin(\frac{1}{x})$ at 0 does not exists, we infer from the Theorem 3.13 that the function is not uniformly continuous on $[0,b]$ for any $b > 0$.
- On the other hand, since $\lim_{x\to 0} x\sin(\frac{1}{x}) = 0$ exists, the function $g(x) := x\sin(\frac{1}{x})$ is uniformly continuous on $[0,b]$ for any $b > 0$, by Theorem 3.13.

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§2 Limit and Continuity Exercises

Problem 2.1. Let $f : \mathbb{R} \xrightarrow{\text{cont.}} \mathbb{R}$ s.t. $f(x) = 0$ for all $x \in \mathbb{Q}$. Then prove that $f(x) = 0$ for all $x \in \mathbb{R}$.

Proof. Suppose $f : \mathbb{R} \xrightarrow{\text{cont.}} \mathbb{R}$. Assume, towards a contradiction, that there exists an $x_0 \in \mathbb{Q}^\complement$ s.t. $f(x_0) \neq 0$, then $d := |f(x_0)| > 0$; take $\varepsilon = d/2 > 0$. As f is continuous on $\mathbb{R}$, there is a $\delta > 0 : |f(x) - f(x_0)| < d/2$ for all $x : \ |x - x_0| < \delta$.

$$\begin{aligned} \text{As } \mathbb{Q} \text{ is dense in } \mathbb{R}, \text{ there is some } x \in \mathbb{Q} : {} & |x - x_0| < \delta \\ \implies {} & |f(x) - f(x_0)| < d/2 \\ \overset{\because x\in\mathbb{Q}}{\implies} {} & 0 < d = |0 - f(x_0)| < d/2. \\ & \text{Absurdity.} \end{aligned}$$

Thus there is no $x \in \mathbb{Q}^\complement$ such that $f(x) \neq 0$, so $f(x) = 0$ for all $x \in \mathbb{R}$. □

Problem 2.2. Let $f, g : \mathbb{R} \xrightarrow{\text{cont.}} \mathbb{R}$ s.t. $f(x) = g(x)$ for all $x \in \mathbb{Q}$. Does it imply $f = g$ on $\mathbb{R}$? Same question for $\mathbb{Q}^\complement$ instead of $\mathbb{Q}$.

Proof. The rationals are dense in $\mathbb{R}$, so for every $c \in \mathbb{R}$ there exists a sequence of rationals $(x_n)_{n\geq 1}$ converging to c. f, g continuous on $\mathbb{R}$ implies that $f(x_n) \to f(c)$ and $g(x_n) \to g(c)$.

But $f(x) = g(x)$ for all rational x, so $f(x_n) = g(x_n)$ for all $n \in \mathbb{Z}^+$, whence

$$f(c) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = g(c) \ \ \forall c \in \mathbb{R}.$$

Here we have only used the fact that the rationals are dense in $\mathbb{R}$. As the irrationals are also dense in $\mathbb{R}$, the same result holds if we replace $\mathbb{Q}$ with $\mathbb{Q}^\complement$. □

Problem 2.3. Let $f, g : \mathbb{R} \xrightarrow{\text{cont.}} \mathbb{R}$ s.t. $f(x) = g(x)$ for all $x \in D$ where D is a dense subset of $\mathbb{R}$. Does it imply $f = g$ on $\mathbb{R}$?

Proof. Yes. As D is dense in $\mathbb{R}$, for every $c \in \mathbb{R}$ there exists a sequence $(x_n)_{n\geq 1}$ of elements in D converging to c. f, g continuous on $\mathbb{R}$ implies that $f(x_n) \to f(c)$ and $g(x_n) \to g(c)$.

But $f(x) = g(x)$ for all $x \in D$, so $f(x_n) = g(x_n)$ for all $n \in \mathbb{Z}^+$, whence

$$f(c) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = g(c) \ \ \forall c \in \mathbb{R}.$$

□

Problem 2.4. Let $f, g : \mathbb{R} \xrightarrow{\text{cont.}} \mathbb{R}$ s.t. $f\left(\dfrac{m}{2^n}\right) = 0$ for all $m, n \in \mathbb{Z}$. Does it imply $f = 0$ on $\mathbb{R}$?

Proof. To show $f = 0$ on $\mathbb{R}$ it suffices to show that the set of dyadic rationals

$S = \left\{ \frac{m}{2^n} : m, n \in \mathbb{Z}^+ \right\}$ is dense in $\mathbb{R}$. For any $x \in \mathbb{R}$,

$$\begin{aligned} & \frac{\lceil 2^n x \rceil}{2^n} \leq \frac{2^n x}{2^n} = x < \frac{\lceil 2^n x \rceil + 1}{2^n} && \forall n \in \mathbb{Z} \\ \implies & 0 \leq x - \frac{\lceil 2^n x \rceil}{2^n} < \frac{1}{2^n} = \varepsilon \\ \implies & \left| x - \frac{\lceil 2^n x \rceil}{2^n} \right| < \varepsilon. \end{aligned}$$

Thus, for every $x \in \mathbb{R}$ there is a sequence $\left(\frac{\lceil 2^n x \rceil}{2^n} \right)_{n \geq 1}$ of dyadic rationals converging to x. Thus, S is dense in $\mathbb{R}$, which, together with $f\left(\frac{m}{2^n}\right) = 0$ for all integers m, n, implies that $f = 0$ on $\mathbb{R}$. □

Problem 2.5. Let $f, g : \mathbb{R} \xrightarrow{\text{cont.}} \mathbb{R}$ s.t. $f\left(m + n\sqrt{2}\right) = 0$ for all $m, n \in \mathbb{Z}$. Does it imply $f = 0$ on $\mathbb{R}$?

Proof. To show $f = 0$ on $\mathbb{R}$ it suffices to show that $S = \left\{ m + n\sqrt{2} : m, n \in \mathbb{Z}^+ \right\}$ is dense in $\mathbb{R}$. Now $(S, +, \cdot)$ is a ring. Every additive subgroup of $\mathbb{R}$ with arbitrarily small positive elements, i.e with limit point 0, is dense in $\mathbb{R}$. As $(S, +)$ is a subgroup of $\mathbb{R}$, we only need to show that 0 is a limit point of S.

Note that $(S, +, \cdot)$ is a ring, so it is closed under multiplication and addition.

$$\begin{aligned} & 0 < \sqrt{2} - 1 = -1 + \sqrt{2} < -1 + \frac{3}{2} = \frac{1}{2} \\ \implies & 0 < (-1 + \sqrt{2})^n < \frac{1}{2^n} = \varepsilon && \forall n \in \mathbb{Z}^+ \\ \implies & \left| (-1 + \sqrt{2})^n \right| < \varepsilon \\ \implies & \lim_{n \to \infty} (-1 + \sqrt{2})^n = 0. \end{aligned}$$

Which proves that $f = 0$ on $\mathbb{R}$ as $f\left(m + n\sqrt{2}\right) = 0$ for all integers m, n. □

Problem 2.6. Let $f, g : \mathbb{R} \xrightarrow{\text{cont.}} \mathbb{R}$ s.t. $f(m + n\alpha) = 0$ for all $m, n \in \mathbb{Z}$ where $\alpha \in \mathbb{Q}^{\complement}$. Does it imply $f = 0$ on $\mathbb{R}$?

Proof. To show $f = 0$ on $\mathbb{R}$ it suffices to show that $S = \left\{ m + n\alpha : m, n \in \mathbb{Z}^+ \right\}$ is dense in $\mathbb{R}$.

$$n\alpha = \lceil n\alpha \rceil + \overbrace{\{n\alpha\}}^{\text{fractional part}},$$

so if we can show that $T = \{\{n\alpha\} : n \in \mathbb{Z}\}$ is dense in $\mathbb{R}$ then we'll have proved that S is dense in $\mathbb{R}$. Now $(T, +, \cdot)$ is a ring. So, as in the preceding problem, we only need to show that 0 is a limit point of T.

Now for each $k = 0, 1, \ldots, n \in \mathbb{Z}^+$ we have $\{k\alpha\} \in [0, 1)$. Write

$$[0, 1) = \left[0, \frac{1}{n}\right) \cup \left[\frac{1}{n}, \frac{2}{n}\right) \cup \cdots \cup \left[\frac{n-1}{n}, 1\right)$$

as the union of n subintervals of length $\frac{1}{n}$. But k can take $n+1$ possible values, so $\{k\alpha\}$ can take $n+1$ values in the interval $[0,1)$. Thus, by the Pigeonhole Principle, there exist $i,j \in \mathbb{Z} : 0 \le i < j \le n$ such that $\{i\alpha\}$, $\{j\alpha\}$ lie in the same interval. Thus,

$$\begin{aligned} & |\{(j-i)\alpha\}| = |\{j\alpha\} - \{i\alpha\}| < \frac{1}{n} = \varepsilon \qquad\qquad \forall n \in \mathbb{Z}^+ \\ \implies & |\{n\alpha\}| < \varepsilon \\ \implies & \lim_{n\to\infty} \{n\alpha\} = 0. \end{aligned}$$

Which proves that $f = 0$ on $\mathbb{R}$ as $f(m+n\alpha) = 0$ for all integers m, n. □

Problem 2.7. Give examples of $f, g : \mathbb{R} \to \mathbb{R}$ s.t. f, g are discontinuous at a pt. x_0 but

(i) $f+g$

(ii) $f-g$

(iii) fg

is continuous at x_0.

Proof. (i) $f = \operatorname{sgn},\ g = -\operatorname{sgn}$ are discontinuous at 0 but $f+g=0$ is continuous at 0 (in fact everywhere).

(ii) $f(x) = \begin{cases} 0, \ x \notin \mathbb{Q} \\ 1, \ x \in \mathbb{Q} \end{cases}$, $g(x) = \begin{cases} -1, \ x \notin \mathbb{Q} \\ 0, \ x \in \mathbb{Q} \end{cases}$ are comtinuous nowhere

but $f-g=1$ is continuous everywhere.

(iii) $f(x) = \begin{cases} 1, \ x > 1 \\ 0, \ x \le 1 \end{cases}$, $g(x) = \begin{cases} 0, \ x > 1 \\ 1, \ x \le 1 \end{cases}$ is discontinuous at 1 but $fg = 0$ is

continuous at 1 (in fact everywhere).

□

Problem 2.8. Let $f, g : \mathbb{R} \to \mathbb{R}$ s.t. f, g are continuous at x_0. Show that $f+g, f-g, fg$ are continuous at x_0. If $g \ne 0$ then show that f/g is continuous at x_0.

Proof. Let (a_n) be a sequence in $\mathbb{R}$ converging to x_0. Then, as f, g are continuous, $f(a_n) \to f(x_0),\ g(a_n) \to g(x_0)$. Thus from the properties of convergent sequences,

$$(f+g)(a_n) \to (f+g)(x_0),\ (f-g)(a_n) \to (f-g)(x_0),$$

$$(fg)(a_n) \to (fg)(x_0), \text{ and if } g \ne 0,\ (f/g)(a_n) = (f/g)(x_0).$$

And sequential continuity is equivalent to $\varepsilon - \delta$ continuity. □

Problem 2.9. (a) Let $f, g, h : \mathbb{R} \to \mathbb{R}$ s.t.

$$f(x) = \begin{cases} 0, \ x \notin \mathbb{Q} \\ 1, \ x \in \mathbb{Q} \end{cases}, \quad g(x) = \begin{cases} 0, \ x \notin \mathbb{Q} \\ x, \ x \in \mathbb{Q} \end{cases}, \quad h(x) = \begin{cases} 0, \ x \notin \mathbb{Q} \\ \frac{1}{q}, \ x = \frac{p}{q} \in \mathbb{Q} \setminus \{0\} \\ 1, \ x = 0 \end{cases}.$$

Show that f is continuous nowhere, g is continuous only at 0 and h is continuous only at irrational points.

(b) Prove that

$$\lim_{m\to\infty} (\cos(\pi x))^{2m}$$

exists for all $x \in \mathbb{R}$ and

$$\lim_{m\to\infty} (\cos(\pi x))^{2m} = \begin{cases} 1, & x \in \mathbb{Z} \\ 0, & x \notin \mathbb{Z} \end{cases} .$$

(c) Prove that

$$\lim_{n\to\infty} \left(\lim_{m\to\infty} (\cos(n!\pi x))^{2m} \right)$$

exists for all $x \in \mathbb{R}$ and

$$\lim_{n\to\infty} \left(\lim_{m\to\infty} (\cos(n!\pi x))^{2m} \right) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases} .$$

Also discuss the continuity of the limit functions.

Proof. (a) Dirichlet's function f is continuous nowhere. For, let $x \in \mathbb{R}$. If $x \in \mathbb{Q}$, then $f(x) = 1$. Choose $\varepsilon = 1/2$. Then for any $\delta > 0$ we can always find an irrational x' (irrationals are dense in $\mathbb{R}$) s.t.

$$|x' - x| < \delta \implies |f(x') - f(x)| = |0 - 1| \geq 1/2 = \varepsilon.$$

Similarly, if $x \notin \mathbb{Q}$ then $f(x) = 0$ so picking $\varepsilon = 1/2$ again, for any $\delta > 0$ we can always find a rational x' (rationals are dense in $\mathbb{R}$) s.t.

$$|x' - x| < \delta \implies |f(x') - f(x)| = |1 - 0| \geq 1/2 = \varepsilon.$$

Thus, f is continuous nowhere.

The function $g(x) = xf(x)$ is only continuous at 0. Let $x \in \mathbb{R} \setminus \{0\}$. If $x \in \mathbb{Q}$, pick some $\varepsilon = |x| > 0$. Then for any $\delta > 0$ we can always find an irrational x' s.t.

$$|x' - x| < \delta \implies |g(x') - g(x)| = |0 - x| = |x| \geq |x| = \varepsilon.$$

Similarly, if $x \notin \mathbb{Q}$, pick some $\varepsilon = |x| > 0$. Then for any $\delta > 0$ we can always find a rational $x' > |x|$ s.t.

$$|x' - x| < \delta \implies |g(x') - g(x)| = |x' - 0| = x' \geq |x| = \varepsilon.$$

Now let $x = 0$. Let $\varepsilon > 0$. Choose $\delta = \min(\varepsilon, 1) > 0$. For any $x' \in (-\delta, \delta) \cap \mathbb{Q}$ we have $|g(x')| = |x'| < \delta < \varepsilon$, and if $x' \in (-\delta, \delta) \cap \mathbb{Q}^{\complement}$ then we trivially have $|g(x')| = 0 < \varepsilon$. Thus for any $\delta > 0$, we have

$$|x'| < \delta \implies |g(x')| < \varepsilon.$$

So g is continuous only at 0.

Thomae's function h is continuous only at the irrationals. Let $c = \frac{m}{n} \in \mathbb{Q}$ with $\gcd(m, n) = 1$. There exists a sequence (x_n) of irrational numbers in $\mathbb{R}$ converging to c. Hence, $h(x_n) = 0$ while $h(c) = \frac{1}{n}$. This shows that h is discontinuous at c. Now let c be irrational and let $\varepsilon > 0$ be arbitrary. Let $N \in \mathbb{N}$ be such that $\frac{1}{N} < \varepsilon$. In the interval $(c - 1, c + 1)$, there are only a finite number of rationals $\frac{m}{n}$ with $n < N$, otherwise we can create a sequence

$\frac{m_k}{n_k}$ with $n_k < N$, all the rationals $\frac{m_k}{n_k}$ distinct and thus necessarily $\frac{m_k}{n_k}$ is unbounded. Hence, there exists $\delta > 0$ such that the interval $(c-\delta, c+\delta)$ contains only rational numbers $x = \frac{m}{n}$ with $n > N$. Hence, if $x = \frac{m}{n} \in (c-\delta, c+\delta)$ then $h(x) = \frac{1}{n} < \frac{1}{N}$ and therefore $|h(x) - h(c)| = \frac{1}{n} < \frac{1}{N} < \varepsilon$. On the other hand, if $x \in (c-\delta, c+\delta)$ is irrational then $|h(x) - h(c)| = |0-0| < \varepsilon$. This proves that f is continuous at c.

(b) For any $m \in \mathbb{Z}$, we have $0 \leq \cos(\pi x)^{2m} \leq 1$. Now, $\cos(\pi x) = \pm 1$ for integer x, so that $\cos(\pi x)^{2m} = 1$. If x is not an integer, we have

$$0 \leq \cos(\pi x)^{2m} < 1 \implies 0 \leq \cos(\pi x)^{2(m+1)} < \cos(\pi x)^2 < 1$$

so that for $x \notin \mathbb{Z}$ the sequence $\left(\cos(\pi x)^{2m}\right)_{m \in \mathbb{Z}^+}$ is decreasing and bounded below by 0, so

$$\lim_{m\to\infty} (\cos(\pi x))^{2m} = \begin{cases} 1, & x \in \mathbb{Z} \\ 0, & x \notin \mathbb{Z} \end{cases}.$$

(c) Let $p = n!x$. Then the inner limit is 1 iff p is integer, and p is an integer iff

$$x = \frac{a}{n!},\ a \in \mathbb{Z}.$$

As $n \to \infty$, x ranges over all possible rational values. So the double limit is 1 iff $x \in \mathbb{Q}$. Likewise, it is 0 iff $x \notin \mathbb{Q}$. Now, let

$$f(x) = \lim_{n\to\infty} \left(\lim_{m\to\infty} (\cos(n!\pi x))^{2m} \right) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases},$$

then f is another form of Dirichlet's function which, as we've proved above, is continuous nowhere.

□

Problem 2.10. Let S be closed in $\mathbb{R}$ and $f : S \xrightarrow{\text{cont.}} \mathbb{R}$. Let $A = \{x \in S : f(x) = 0\}$. Show that A is closed in $\mathbb{R}$.

Proof. f is continuous so K closed in $\mathbb{R} \implies f^{-1}(K)$ closed in S, and if K is closed in a closed subset of $\mathbb{R}$ then K is closed in $\mathbb{R}$, so K is closed in $\mathbb{R}$ as S is closed in $\mathbb{R}$. Now, $\{0\}$ is closed in $\mathbb{R}$ so $f^{-1}(\{0\}) = A$ is closed in $\mathbb{R}$, and thus A is closed in $\mathbb{R}$. □

Problem 2.11. Let $f, g : \mathbb{R} \xrightarrow{\text{cont.}} \mathbb{R}$.

(i) Show that $G = \{x \in \mathbb{R} : f(x) > 0\}$ is open in $\mathbb{R}$.

(ii) Show that $F = \{x \in \mathbb{R} : f(x) = 0\}$ is closed in $\mathbb{R}$.

(iii) Show that $F = \{x \in \mathbb{R} : f(x) \leq 0\}$ is closed in $\mathbb{R}$.

(iv) Show that $G = \{x \in \mathbb{R} : f(x) > g(x) + c\}$ is open in $\mathbb{R}$ for fixed $c \in \mathbb{R}$.

(v) Show that $F = \{x \in \mathbb{R} : f(x) \leq g(x) + c\}$ is closed in $\mathbb{R}$ for fixed $c \in \mathbb{R}$.

(vi) Show that $S_1 = \{x \in \mathbb{R} : e^x > 1\}$, $S_2 = \{x \in \mathbb{R} : \sin(x) < 1\}$ are open in $\mathbb{R}$ whereas $S_3 = \{x \in \mathbb{R} : e^x = cosx\}$, $S_4 = \{x \in \mathbb{R} : \sin(x) + 2\cos(x) \geq 2\}$ are closed in $\mathbb{R}$.

One can replace the $>$ by $<$ and $\leq$ by $\geq$ in the above results.

Proof.
1. $(0, \infty)$ is open in $\mathbb{R}$ so $G = f^{-1}((0, \infty))$ is open in $\mathbb{R}$.
2. $\{0\}$ is closed in $\mathbb{R}$ so $F = f^{-1}(\{0\})$ is closed in $\mathbb{R}$.
3. $(-\infty, 0]$ is closed in $\mathbb{R}$ so $F = f^{-1}((-\infty, 0])$ is closed in $\mathbb{R}$.
4. $h = f - g$ is continuous and $G = \{x \in \mathbb{R} : f(x) > g(x) + c\}$
$$= \{x \in \mathbb{R} : h(x) > c\}$$
so $G = h^{-1}((c, \infty))$ is closed in $\mathbb{R}$.
5. $h = f - g$ is continuous and $F = \{x \in \mathbb{R} : f(x) \leq g(x) + c\}$
$$= \{x \in \mathbb{R} : h(x) \leq c\}$$
so $F = h^{-1}((-\infty, c])$ is closed in $\mathbb{R}$.
6. $f_1(x) = e^x, f_2(x) = -\sin(x), f_3(x) = e^x - \cos(x), f_4(x) = \sin(x) + 2\cos(x)$ are continuous functions;
$$S_1 = \{x \in \mathbb{R} : f_1(x) > 1\},\ S_2 = \{x \in \mathbb{R} : f_2(x) > -1\}$$
so they are open in $\mathbb{R}$ whereas
$$S_3 = \{x \in \mathbb{R} : f_3(x) = 0\},\ S_4 = \{x \in \mathbb{R} : f_4(x) \geq 2\}$$
so they are closed in $\mathbb{R}$.

□

Problem 2.12. Let $f : \mathbb{R} \xrightarrow{\text{cont.}} \mathbb{R}$ and $f(x_0) > c$ for some $x_0,\ c \in \mathbb{R}$. Show that there exists a neighbourhood U of x_0 s.t. $f(x) > c\ \ \forall x \in U$.

Proof. Choose an $\varepsilon = f(x_0) - c > 0$. Then, as f is continuous, there exists a $\delta > 0 :$

$$\begin{aligned} &|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon \\ &\implies f(x_0) - \varepsilon < f(x) < f(x_0) + \varepsilon \\ &\implies f(x_0) - (f(x_0) - c) < f(x) \implies f(x) > c. \\ &\text{Set } V_\delta(x_0) = \{x \in \mathbb{R} : |x - x_0| < \delta\}. \end{aligned}$$

Thus, there exists a $\delta > 0 : x \in V_\delta(x_0) \implies f(x) > c$. □

Problem 2.13. Let $f : [a, b] \xrightarrow{\text{cont.}} \mathbb{R}$ and $f(x) > 0\ \ \forall x \in [a, b]$. Show that there exists a $c > 0$ s.t. $f(x) \geq c\ \ \forall x \in [a, b]$.

Proof. Let $c = \min(\{f(x) : x \in [a, b]\}) > 0$. Such a c exists in $[a, b]$ as $[a, b]$ is compact: $f([a, b])$ is the continuous image of a compact set so it is compact, thus bounded so admitting an infimum and closed so that the infimum is in the set i.e. minimum. Thus, $f(x) \geq c$ for all $x \in [a, b]$. □

Problem 2.14. Let $f : [a,b] \xrightarrow{\text{cont.}} \mathbb{R}$ s.t. for each $x \in [a,b]$ there exists a $y \in [a,b]$ with

$$|f(y)| \leq \frac{1}{2}|f(x)|.$$

Show that there exists a $c \in [a,b]$ s.t. $f(c) = 0$.

> *Proof.* Construct the sequence $(x_n)_{n\geq 1}$ of elements in $[a,b]$ iteratively by setting $x_1 = b$ and for each $x_n = x \in [a,b]$ letting $x_{n+1} = y \in [a,b]$ such that
>
> $$|f(x_{n+1})| \leq \frac{1}{2}|f(x_n)|,\ n > 1.$$
>
> Then clearly,
>
> $$|f(x_n)| \leq \frac{1}{2^n}|f(b)| \implies |f(x_n)| \to 0 \text{ as } n \to \infty.$$
>
> Now, $[a,b]$ is compact so it is sequentially compact. Thus there exists a convergent subsequence (x_{n_k}) of (x_n). Suppose $x_{n_k} \to c$. Then $f(x_{n_k}) \to f(c)$. But $f(x_{n_k}) \to 0$, so $f(c) = 0$. □

Problem 2.15. In the following either give an example of a continuous function f such that $f(S) = T$ or explain that there can be no such f :

we use the fact that the continuous image of a compact (connected) set is compact (connected)

(i) $S = (0,1),\ T = (0,1]$.
Ans.: Yes. $f = \begin{cases} 2x,\ x < 1/2 \\ 2(1-x),\ x \geq 1/2 \end{cases}$.

(ii) $S = (0,1),\ T = [0,1]$.
Ans.: Yes. $f = \begin{cases} 0,\ x < 1/4 \\ 2(x - 1/4),\ x \in [1/4, 3/4] \\ 1,\ x > 3/4 \end{cases}$.

(iii) $S = (0,1),\ T = (1,2) \cup (2,3)$.
Ans.: No. S is connected but T is not connected.

(iv) $S = (-1,1),\ T = \mathbb{R}$.
Ans.: Yes. $f = \dfrac{x}{1-x^2}$ which is continuous on $(-1,1)$.

(v) $S = \mathbb{R},\ T = (-1,1)$.
Ans.: Yes. $f = \dfrac{2}{\pi}\arctan(x)$ which is continuous on $\mathbb{R}$.

(vi) $S = [0,1],\ T = \mathbb{R}$.
Ans.: No. S is compact but T is not compact.

(vii) $S = [0,1],\ T = \mathbb{Q}$.
Ans.: No. S is compact but T is not compact.

(viii) $S = \mathbb{R},\ T = \mathbb{Q}$.
Ans.: No. S is connected but T is not connected.

(ix) $S = (0,1) \cup (2,3),\ T = \{0,3\}$.
Ans.: No. S is connected but T is not connected.

(x) $S = \left\{\frac{1}{n} : n \in \mathbb{Z}^+\right\} \cup \{0\},\ T = \left\{\frac{1}{2n} : n \in \mathbb{Z}^+\right\}$.
Ans.: No. S is compact but T is not compact.

(xi) $S = \left\{\frac{1}{n^2} : n \in \mathbb{Z}^+\right\} \cup \{0\},\ T = \mathbb{Z}^+$.
Ans.: No. S is compact but T is not compact.

Problem 2.16. Let $f : [a, b] \xrightarrow{\text{cont.}} \mathbb{R}$. Define $g : [a, b] \to \mathbb{R}$ by

$$g(x) = \max_{a \le y \le x} f(y),\ a \le x \le b.$$

Show that g is continuous on $[a, b]$.

Proof. As $[a, b]$ is compact, f attains its maximum on $[a, b]$ and so g is well-defined. Sps $x_1,\ x_2 \in [a, b]$ with $x_2 > x_1$. Then,

$$g(x_2) = \max_{a \le y \le x_2} f(y) = \max\left(\max_{a \le y \le x_1} f(y),\ \max_{x_1 \le y \le x_2} f(y)\right)$$

$$= \max\left(g(x_1),\ \max_{x_1 \le y \le x_2} f(y)\right) \ge g(x_1).$$

So g is increasing on $[a, b]$. Also clearly at any point $x_0 \in [a, b]$ we have $g(x_0) \ge f(x_0)$. So we have two cases: either $g(x_0) > f(x_0)$ or $g(x_0) = f(x_0)$.

First case: $g(x_0) > f(x_0)$

We know that there exists a $\delta > 0$ s.t $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon_0$ (since f is continuous), with $\varepsilon_0 := g(x_0) - f(x_0) > 0$. So we have

$$|x - x_0| < \delta \Rightarrow f(x) < f(x_0) + \varepsilon_0 = g(x_0).$$

So for $x \in [a, b]$ with $|x - x_0| < \delta$ we have $|g(x) - g(x_0)| = 0$.

Then for $x \in [a, b]$ we have:

$$\forall \varepsilon > 0 : |x - x_0| < \delta \Rightarrow |g(x) - g(x_0)| = 0 < \varepsilon.$$

By which we have the continuity of g.

Second case: $g(x_0) = f(x_0)$

For $x \in [a, x_0]$ we have:

$$f(x) \le g(x) \Rightarrow -g(x) \le -f(x) \Rightarrow g(x_0) - g(x) \le g(x_0) - f(x)$$

$$\Rightarrow 0 \le g(x_0) - g(x) \le f(x_0) - f(x) \Rightarrow |g(x) - g(x_0)| \le |f(x) - f(x_0)|.$$

Hence with continuity of f we get the continuity of g.

For $x \in [x_0, b]$ we have $g(x) = f(s)$ for some $s \in [x_0, x]$. With continuity of f we have:

$$\forall \varepsilon > 0\ \exists \delta > 0 : |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.$$

Hence

$$x \in [x_0, x_0 + \delta) \Rightarrow |f(s) - g(x_0)| < \varepsilon \Rightarrow |g(x) - g(x_0)| < \varepsilon.$$

By which we have, again, the continuity of g. □

Problem 2.17. A function f defined on an interval I is said to be convex on I iff

$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y), \;\; \forall x, y \in I, \; t \in [0,1].$$

Prove that if f is convex on an open inteval then f is continuous. Verify whether the result is true for arbitrary intervals also.

Proof. Suppose $I = (a, b)$. Pick any two points $c, d \in (a, b) : \; c < d$.
Let $\eta > 0 : \;\; \eta < \dfrac{d-c}{2}$. Consider $x, y \in [c + \eta, d - \eta] : \;\; x < y$. As f is convex on (a, b), we have that f is bounded on $[c, d] \subset (a, b)$ using the inequality,

$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y), \;\; \forall t \in [0,1].$$

For, if $z \in (c, d)$ then taking $t = \dfrac{z-c}{d-c} \in (0,1)$ we have $z = t(d-c)+c = (1-t)c+td$, and letting $M = \max(f(c), f(d))$, we have

$$f(z) = f((1-t)c + td) \leq (1-t)f(c) + tf(d) \leq (1-t)M + tM = M.$$

If $z \in \left(\dfrac{c+d}{2}, d\right)$ we have

$$\frac{c+d}{2} = (1-t)c + tz, \; t = \frac{\frac{c+d}{2} - c}{z - c} = \frac{d-c}{2(z-c)} \in (0,1)$$

so that

$$f\left(\frac{c+d}{2}\right) = f((1-t)c + tz) \leq (1-t)f(c) + tf(z) \leq f(c) + f(z)$$

$$\implies f(z) \geq f\left(\frac{c+d}{2}\right) - f(c).$$

If $z \in \left(c, \dfrac{c+d}{2}\right)$ we have

$$\frac{c+d}{2} = (1-t)z + td, \; t = \frac{\frac{c+d}{2} - z}{d - z} = \frac{c+d-2z}{2(d-z)} \in (0,1)$$

so that

$$f\left(\frac{c+d}{2}\right) = f((1-t)z + td) \leq (1-t)f(z) + tf(d) \leq f(z) + f(d)$$

$$\implies f(z) \geq f\left(\frac{c+d}{2}\right) - f(d).$$

Letting

$$m = \left(f(c), f(d), f\left(\frac{c+d}{2}\right), f\left(\frac{c+d}{2}\right) - f(c), f\left(\frac{c+d}{2}\right) - f(d) \right)$$

we have $m \leq f(z) \leq M$, $\forall z \in [c, d]$. Thus f is bounded on $[c, d]$, i.e., $|f(x)| \leq K$ for some $K > 0$. We can then write,

$$x = (1-t)c + ty, \ t = \frac{x-c}{y-c} \in (0,1).$$

Then,

$$\begin{aligned} f(x) \leq (1-t)f(c) + tf(y) \implies f(x) - f(y) &\leq (1-t)(f(c) - f(y)) \\ &= \frac{y-x}{y-c}(f(c) - f(y)) \\ \implies f(x) - f(y) \leq \frac{y-x}{y-c}(f(c) - f(y)) &\leq \frac{y-x}{\eta}|f(c) - f(y)| \\ &\leq \frac{2M}{\eta}(y-x). \end{aligned}$$

Similarly, $f(y) - f(x) \leq \frac{2M}{\eta}(y-x)$ so that

$$|f(y) - f(x)| \leq \frac{2M}{\eta}|y-x|, \ \forall x, y \in [c+\eta, d-\eta].$$

As c, d, η are arbitrary, it follows that f is continuous on (a, b). Because for any $\varepsilon > 0$, we can pick a $\delta = \dfrac{\eta}{2M}\varepsilon > 0$ s.t.

$$|y-x| < \delta \implies |f(y) - f(x)| \leq \frac{2M}{\eta}|y-x| < \frac{2M}{\eta}\delta = \varepsilon.$$

If I is a general interval and f is convex on I, then f need not be continuous. For example, define $f : [0,1) \to \mathbb{R}$ by

$$f = \begin{cases} -\sqrt{x}, \ x > 0 \\ 1, \ x = 0 \end{cases}$$

then clearly f is convex on $[0,1)$ but discontinuous at $x = 0$. □

Problem 2.18. Let $f : (a,b) \xrightarrow{\text{cont.}} \mathbb{R}$. Show that for each $x \in (a,b)$ there exists a neighbourhood $(x - \delta_x, x + \delta_x) = V_x$ such that f is bounded on V_x. Note that f may not be bounded on (a,b).

Proof. By definition, f is continuous at $x \in (a,b)$ iff for all $\varepsilon > 0$ there exists $\delta_x > 0$:

$$|x' - x| < \delta \implies |f(x') - f(x)| < \varepsilon,$$

$$x' \in (x - \delta_x, x + \delta_x) \implies f(x) - \varepsilon < f(x') < f(x) + \varepsilon,$$

$$x' \in V_x \implies |f(x')| \leq \max(|f(x) - \varepsilon|, |f(x) + \varepsilon|).$$

Which is what was to be shown. □

Problem 2.19. Let $f : [a, b] \to \mathbb{R}$ s.t. for each $x \in [a, b]$ there exists a neighbourhood $(x - \delta_x, x + \delta_x) = V_x$ s.t. f is bounded on V_x. Show that f is bounded on $[a, b]$. Note that f may not be continuous on $[a, b]$.

Proof. $[a, b]$ is compact so it is sequentially compact i.e. there exists a sequence $(x_n) \subset [a, b]$ with a subsequence (x_{n_k}) s.t. $x_{n_k} \to \ell \in [a, b]$. Assume towards a contradiction that there exists an $x \in [a, b]$ s.t. f is unbounded. Then $f(x_n) > n$ for every $n \in \mathbb{Z}^+$. From the problem there exists δ_ℓ corresponding to ℓ s.t. f is bounded on $V_\ell = (\ell - \delta_\ell, \ell + \delta_\ell)$. Choose n large enough s.t. $x_n \in V_\ell$. Then $f(x_n) > n$ but f is bounded on V_ℓ. Absurdity. Thus, f cannot be unbounded at any point $x \in [a, b]$. □

Problem 2.20. Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$.

(i) Let f be continuous on A. If $(x_n)_{n\geq 1}$ is a convergent sequence in A then show that $(f(x_n))_{n\geq 1}$ is also convergent. Verify whether *convergent* can be replaced by *Cauchy* or *bounded*.

(ii) Let f be such that if $(x_n)_{n\geq 1}$ is a Cauchy sequence then $(f(x_n))_{n\geq 1}$ is also Cauchy in A. Hence show that f is continuous on A.

Proof. (i) Sps f is continuous at $x \in A$ and suppose $(x_n) \subset A : x_n \to x$. Then it suffices to show that $f(x_n) \to f(x)$. For any $\varepsilon > 0$, by definition, there exists $\delta > 0 :$

$$x' \in A : |x' - x| < \delta \implies |f(x') - f(x)| < \varepsilon.$$

But as $\delta > 0$ we also have

$$|x_n - x| < \delta,\ n \geq n_0$$

for some $n_0 \in \mathbb{Z}^+$. Thus,

$$|f(x_n) - f(x)| < \varepsilon,\ n \geq n_0.$$

As $\mathbb{R}$ is a complete metric space, convergent can be replaced by Cauchy as convergence is equivalent to Cauchy convergence in a complete metric space. Also, convergent can be replaced by bounded as convergence implies boundedness.

(ii) We are working over $A \subseteq \mathbb{R}$ so convergence and Cauchy are equivalent. Sps $(x_n) \subset A$ is Cauchy s.t. $x_n \to x$, then from the problem we have that $(f(x_n))$ is Cauchy as well s.t. $f(x_n) \to f(x)$. Assume towards a contradiction that f is discontinuous at x, so there exists $\varepsilon > 0 : \forall \delta > 0 :$

$$|x' - x| < \delta \implies |f(x') - f(x)| \geq \varepsilon.$$

Take $\delta = \dfrac{1}{n} > 0$, then

$$|x_n - x| < \frac{1}{n} = \delta \implies |f(x_n) - f(x)| \geq \varepsilon\ \forall n \in \mathbb{Z}^+$$
$$\implies f(x_n) \not\to f(x).$$

But from the problem $f(x_n) \to x$. Absurdity. Thus, our assumption was wrong and f must be continuous on A.

□

Problem 2.21. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. Suppose f has local maxima at two different points x_1 and x_2. Show that there must be a third point between x_1 and x_2 where f has has a local minimum.

Problem 2.22. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. If f has neither a local maximum nor a local minimum at any interior point then prove that f must be monotonic on $[a, b]$.

Problem 2.23. Let $f : (a, b) \to \mathbb{R}$ be a function such that for each $x \in [a, b]$ there exists a neighbourhood $(x - \delta_x, x + \delta_x) = V_x$ such that f is increasing on V_x. Show that f is an increasing function throughout (a, b).

Problem 2.24. (i). Let $f : [a, b] \to \mathbb{R}$ be monotone increasing and $c \in (a, b)$. Then show that $f(c-)$ and $f(c+)$ exist and

$$f(c-) = \sup(f(x) : a \leq x < c) \text{ and } f(c+) = \inf(f(x) : c < x \leq b).$$

(ii). Let $f : [a, b] \to \mathbb{R}$ be monotone decreasing and $c \in (a, b)$. Then show that $f(c-)$ and $f(c+)$ exist and

$$f(c-) = \inf(f(x) : a \leq x < c) \text{ and } f(c+) = \sup(f(x) : c < x \leq b).$$

(iii) Let D_f denote the set of discontinuities of a monotone increasing or decreasing function f defined on an interval I. Show that f can't have discontinuities of the 2nd kind and D_f is countable.
(iv) Let $S = \{x_1, x_2, \dots\} \subset [a, b]$. Does there exist a function $f : [a, b] \to \mathbb{R}$ such that f is monotone and the set of discontinuities of f is S ?

Proof. (i) For $a \leq x < c$ we have $f(x) \leq f(c)$ as f is increasing. So the set $\{f(x) : a \leq x < c\}$ is bounded above and nonempty as $f(a)$ is in the set. Thus, $M = \sup(f(x) : a \leq x < c) \in \mathbb{R}$ exists. Let $\varepsilon > 0$. Then there exists an element $x_\varepsilon : a \leq x_\varepsilon < c$ s.t. $M - \varepsilon < f(x_\varepsilon) \leq M$. Letting $\delta = c - x_\varepsilon > 0$ we have

$$x \in (c-\delta, c) \cap [a, b] \implies x \in (x_\varepsilon, c) \cap [a, b] \implies M-\varepsilon < f(x_\varepsilon) \leq f(x) < M+\varepsilon$$

$$\implies |f(x) - M| < \varepsilon \; \forall x \in (c - \delta, \delta) \cap [a, b]$$

Thus, $\lim_{x \to c-} f(x) = M = \sup(f(x) : a \leq x < c)$.

(ii) For $a \leq x < c$ we have $f(x) \geq f(c)$ as f is decreasing. So the set $\{f(x) : a \leq x < c\}$ is bounded below and nonempty as $f(a)$ is in the set. Thus, $m = \inf(f(x) : a \leq x < c) \in \mathbb{R}$ exists. Let $\varepsilon > 0$. Then there exists an element $x_\varepsilon : a \leq x_\varepsilon < c$ s.t. $m \leq f(x_\varepsilon) < m + \varepsilon$. Letting $\delta = c - x_\varepsilon > 0$ we have

$$x \in (c-\delta, c) \cap [a, b] \implies x \in (x_\varepsilon, c) \cap [a, b] \implies m-\varepsilon < f(x) \leq f(x_\varepsilon) < m+\varepsilon$$

$$\implies |f(x) - m| < \varepsilon \; \forall x \in (c - \delta, \delta) \cap [a, b]$$

Thus, $\lim_{x \to c-} f(x) = m = \inf(f(x) : a \le x < c)$.

□

Problem 2.25. Let $f : [a, b] \to \mathbb{R}$ be monotone increasing. Then show that f is continuous at a iff $f(a) = \inf(f(x) : a < x \le b)$ and f is continuous at b iff $f(b) = \sup(f(x) : a \le x < b)$. (Similar result holds for monotone decreasing functions).

Problem 2.26. (i) If f is one-one and continuous on $[a, b]$ then prove that f must be strictly monotonic on $[a, b]$. If f is strictly increasing then show that $f^{-1} : [f(a), f(b)] \to [a, b]$ exists, and is strictly increasing and continuous.
(ii) Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. Then show that there is a function g such that $g \circ f = id$ iff f is strictly monotone. Such a function f is called homeomorphism between domain $[a, b]$ and the range $[f(a), f(b)]$.

Problem 2.27. Give an example of a function f defined and strictly increasing on a set S in $\mathbb{R}$ such that f^{-1} is not continuous on $f(S)$.

Problem 2.28. Let f be strictly increasing on a subset S of $\mathbb{R}$. Show that f must be continuous on S if $f(S)$ has one of the following properties:

(i) $f(S)$ is open (ii) $f(S)$ is connected (iii) $f(S)$ is closed.

Problem 2.29. Let $f : [0, 1] \xrightarrow{\text{cont.}} \mathbb{R}$ s.t. for every $y \in \mathbb{R}$ either there is no $x \in [0, 1] : f(x) = y$ or there is exactly one such x. Show that f is strictly monotonic on $[0, 1]$.

Problem 2.30. Let $f : [0, 1] \to \mathbb{R}$ s.t. for every $y \in \mathbb{R}$ either there is no $x \in [0, 1] : f(x) = y$ or there are exactly two such values of $x \in [0, 1] : f(x) = y$. Then,

(i) Prove that f can't be continuous on $[0, 1]$.

(ii) Construct a function f which has the above property.

(iii) Prove that any function with this property has infinitely many discontinuities on $[0, 1]$.

Problem 2.31. Let $f : [0, 1] \to \mathbb{R}$ s.t.

$$f(x) = \begin{cases} 1 - x, \ x \in \mathbb{Q}^{\complement} \\ x, \ x \in \mathbb{Q} \end{cases}$$

then show that

(i) $f(f(x)) = x$ for all $x \in [0, 1]$.

(ii) $f(x) + f(1 - x) = 1$ for all $x \in [0, 1]$.

(iii) f is continuous only at $x = \dfrac{1}{2}$.

(iv) f attains every value between 0 and 1.

(v) $f(x + y) - f(x) - f(y)$ is rational for all $x, y \in [0, 1]$.
This shows that the converse of the intermediate value theorem is not true.

Proof. (i) $f(f(x)) = f(1 - x) = 1 - (1 - x) = x$.

□

Problem 2.32. Use intermediate value theorem for continuous functions to prove the following:

(i) If $n \in \mathbb{Z}^+$ and $a > 0$ then there is exactly one $b \in \mathbb{R} : b^n = a$.

(ii) Let $f(x) = \tan(x)$. Although $f(\pi/4) = 1$ and $f(3\pi/4) = 1$ there is no $x \in [\pi/4, 3\pi/4] : f(x) = 0$. Explain why this does not contradict the intermediate value theorem.

(iii) Let $f : [a, b] \overset{\text{cont.}}{\longrightarrow} \mathbb{R}$. If $f(a) \leq a$ and $f(b) \geq b$ then prove that f has a fixed point in $[a, b]$ i.e. there exists some point $c \in [a, b] : f(c) = c$.

Problem 2.33. Let F be a closed set of real numbers and f be a real-valued function continuous on F. Show that there exists a function g well-defined and continuous on $\mathbb{R}$ s.t. $f(x) = g(x)\ \forall x \in F$. Such a function g is called the extension of f. Show that the extension may not be possible if F is not closed.

Problem 2.34. (a) Let $f : S \subseteq \mathbb{R} \overset{\text{uniform cont.}}{\longrightarrow} \mathbb{R}$. Show that

(i) $(x_n) \subset S$ convergent in $S \implies (f(x_n))$ convergent in $\mathbb{R}$.

(ii) $(x_n) \subset S$ Cauchy in $S \implies (f(x_n))$ Cauchy in $\mathbb{R}$.

(iii) $A \subset S$ bounded $\implies f(A)$ bounded. Give an example to show that if A is not bounded then $f(A)$ need not be bounded.

(b) Let $f : S \subseteq \mathbb{R} \to \mathbb{R}$ s.t. if (x_n) is Cauchy in S then $(f(x_n))$ is Cauchy in $\mathbb{R}$. Verify whether f is uniformly continuous or not.

Problem 2.35. (i) Let $f : S \subseteq \mathbb{R} \to \mathbb{R}$ s.t. $f(x) = x^2$. If $S \subseteq \mathbb{R}$ is bounded then show that f is uniformly continuous on S. What can you say if S is not bounded ?

(ii) Verify whether the following statement is true or false: Let $f : K \subseteq \mathbb{R} \overset{\text{uniform cont.}}{\longrightarrow} \mathbb{R}$. Then there exists a constant $M > 0$ s.t. $|f(x)| \leq M|x|\ \forall x \in \mathbb{R}$.

Problem 2.36. Prove that $f : S \subseteq \mathbb{R} \overset{\text{uniform cont.}}{\longrightarrow} \mathbb{R}$ iff for every pair of sequences $(x_n),\ (y_n) \subset S$ with $\lim(x_n - y_n) = 0$ we have $\lim(f(x_n) - f(y_n)) = 0$.

Problem 2.37. Show that the functions

$$f(x) = \frac{1}{x},\ g(x) = \sin\left(\frac{1}{x}\right)$$

are not uniformly continuous on $(0, \infty)$.

Problem 2.38. (i) Let $f : [a, b] \overset{\text{cont.}}{\longrightarrow} \mathbb{R}$. Then show that f is uniformly continuous on $[a, b]$. *The result holds if* $[a, b]$ *is replaced by a compact set* K.

(ii) Let $f : (a, b) \overset{\text{cont.}}{\longrightarrow} \mathbb{R}$. Then show that f is uniformly continuous iff

$$\lim_{x \to a+} f(x) \text{ and } \lim_{x \to a-} f(x)$$

exist finitely.

(iii) Prove that f is uniformly continuous on (a, b) iff f can be defined at the endpoints a, b s.t. the extended function is continuous on $[a, b]$.

Problem 2.39. Show that $K \subseteq \mathbb{R}$ is compact iff every continuous function $f : K \to \mathbb{R}$ attains its maximum value.

Problem 2.40. Let $f : K \subseteq \mathbb{R} \xrightarrow{\text{cont.}} \mathbb{R}$ be bijective. If K is compact show that f^{-1} is also continuous.

Problem 2.41. Show that if f is continuous on $[0, \infty)$, and uniformly continuous on $[a, \infty)$ for some constant $a > 0$ then f is uniformly continuous on $[0, \infty)$.

Problem 2.42. Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ s.t. for every $\varepsilon > 0$ there exists a function $g_\varepsilon : A \to \mathbb{R}$ s.t. g_ε is uniformly continuous on A and

$$|f(x) - g_\varepsilon(x)| < \varepsilon \ \forall x \in A.$$

Show that f is uniformly continuous on A.

Problem 2.43. Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and $K > 0$ be a constant. If

$$|f(x) - f(y)| < K|x - y|^\alpha \ \forall x \in A$$

then f is said to be Lipschitz of order α.

(i) Show that f is continuous if $\alpha > 0$ and differentiable if $\alpha > 1$.

(ii) Find a Lipschitz function of order 1 for which the derivative does not exist.

(iii) If $\alpha = 1$ then show that f is uniformly continuous.

Problem 2.44. Let $f : \mathbb{R} \to \mathbb{R}$ and t, a be nonzero fixed reals. Define $g, h : \mathbb{R} \to \mathbb{R}$ by $g(x) = f(x + t),\ h(x) = f(ax)$. If f is continuous (uniformly continuous) then show that g, h are continuous (uniformly continuous).

Problem 2.45. Let $f : (a, b) \to \mathbb{R}$ be differentiable and f' be bounded on (a, b). Then show that f is uniformly continuous on (a, b). Find a counterexample to show that the converse need not be true.

Problem 2.46. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy Cauchy's functional equation

$$f(x + y) = f(x) + f(y),\ \forall x, y \in \mathbb{R}.$$

(i) If f is continuous at a point x_0 then show that f is continuous on $\mathbb{R}$ and there exists a constant c s.t.

$$f(x) = cx \ \forall x \in \mathbb{R}.$$

(ii) If f is bounded above on some interval or f is monotonic on $\mathbb{R}$ then there exists a constant c s.t.

$$f(x) = cx \ \forall x \in \mathbb{R}.$$

Problem 2.47. Let $f : \mathbb{R} \to \mathbb{R}$ s.t.

$$f(x + y) = f(x)f(y),\ \forall x, y \in \mathbb{R}.$$

If f is continuous at $x = 0$ then show that f is continuous on $\mathbb{R}$. Show that there exists a constant c s.t.

$$f(x) = e^{cx} \ \forall x \in \mathbb{R}.$$

Problem 2.48. Let $f : (0, \infty) \to \mathbb{R}$ s.t.

$$f(xy) = f(x) + f(y),\ \forall x, y \in \mathbb{R}.$$

If f is continuous at $x_0 \in (0, \infty)$ then show that f is continuous on $\mathbb{R}$. Show that there exists a constant c s.t.

$$f(x) = c \log(x) \ \forall x \in (0, \infty).$$

Problem 2.49. Let $f : (0, \infty) \to \mathbb{R}$ s.t.

$$f(xy) = f(x)f(y),\ \forall x, y \in \mathbb{R}.$$

If f is continuous at $x_0 \in (0, \infty)$ then show that f is continuous on $\mathbb{R}$. Find all such continuous functions.

Problem 2.50. Find all $f : \mathbb{R} \xrightarrow{\text{cont.}} \mathbb{R}$ s.t. $f(x) - f(y)$ is rational for rational $x - y$.

Problem 2.51. For $|q| < 1$ find all functions $f : \mathbb{R} \to \mathbb{R}$ s.t. f is continuous at 0 and satisfies the functional equation

$$f(x) + f(qx) = 0.$$

Problem 2.52. Find all functions $f : \mathbb{R} \to \mathbb{R}$ s.t. f is continuous at 0 and satisfies the functional equation

$$f(x) + f\left(\frac{2}{3}x\right) = 0.$$

Problem 2.53. Find all $f : \mathbb{R} \xrightarrow{\text{cont.}} \mathbb{R}$ satisfying the Jensen equation

$$f\left(\frac{x+y}{2}\right) = \frac{f(x) + f(y)}{2}.$$

Problem 2.54. Find all $f : (a, b) \xrightarrow{\text{cont.}} \mathbb{R}$ satisfying the Jensen equation

$$f\left(\frac{x+y}{2}\right) = \frac{f(x) + f(y)}{2}.$$

Problem 2.55. If $f : \mathbb{R} \to \mathbb{R}$ be a function s.t. either $f(x-)$ or $f(x+)$ exists finitely then show that the set of discontinuities of f is countable. The result holds even if the limit exists infinitely.

§3 Differentiability

Notes from the professor are appended in the following pages.

[1]

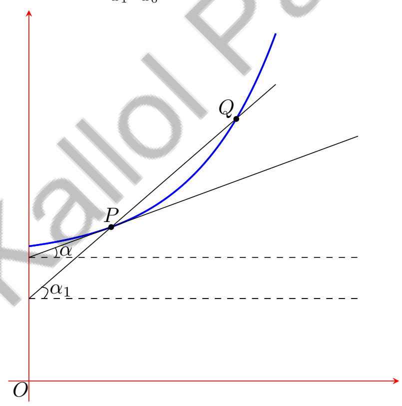
DIFFERENTIABILITY OF REAL VALUED FUNCTION

PROF KALLOL PAUL

We begin with the definition of tangent to a curve $y = f(x)$ at a point P. By a curve we mean a continuous function $f : [a, b] \to \mathbb{R}$.

Definition 0.1 (**Tangent**)**.** Let $y = f(x)$ be a given curve and $P(x_0, y_0)$ be any point on it. Consider another point $Q(x_1, y_1)$ near to P on the curve. Then draw a straight line, known as secant of the curve, passing through points P and Q. Let us now move the point Q towards P along the curve. If the limiting position of the secant as Q approaches P exists then the limiting position of the secant is known as the tangent to the curve $y = f(x)$ at P. The existence of such a tangent to the curve at a point is not always guaranteed.

Let α_1 be the angle made by the secant PQ with the positive x-axis and α be the angle made by the tangent at P with the positive x-axis. Then $\tan \alpha_1 = \frac{y_1 - y_0}{x_1 - x_0}$. As Q approaches P along the curve we get α_1 approaches α and so $\lim_{Q \to P} \frac{y_1 - y_0}{x_1 - x_0} = \lim_{\alpha_1 \to \alpha} \tan \alpha_1 = \tan \alpha$.



Next we define derivative of a function $y = f(x)$ at a point x_0.

Definition 0.2 (**Derivative**)**.** Let $f : S \subset R \to R$ be a function and $x_0 \in S$ be a limit point of S. Then the function f is said to have a derivative at x_0 if there exists a real number L such that

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = L.$$

In other words f is said to be differentiable at x_0 if there exists a real number L such that for any given $\epsilon > 0$ there exists $\delta > 0$(depending on x_0 and ϵ) that satisfies the following:

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - L \right| < \epsilon \ \forall \ x \in (x_0 - \delta, x_0 + \delta) \cap S, x \neq x_0.$$

[1]For any further readings please see books by Rudin, Apostol or Bartle and Sherbert.

In such a case we say f is derivable at x_0 or f is differentiable at x_0 and we write $f'(x_0) = L$.
Observe that the existence of the limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

is equivalent to the existence of the limit

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

Let us note that these two definitions are equivalent in the sense that if f is derivable at a point x_0 then the tangent to the function f exists at the point $P(x_0, y_0)$,

$$\lim_{Q \to P} \frac{y_1 - y_0}{x_1 - x_0} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

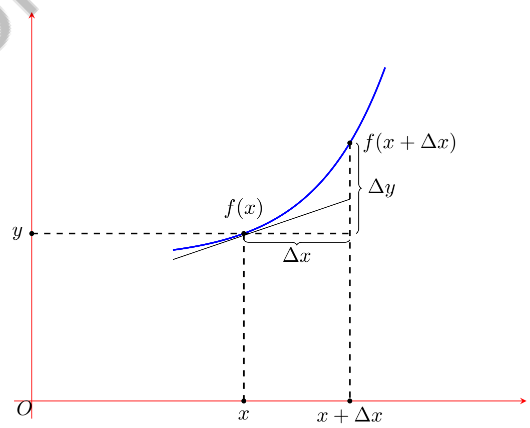
The first definition is geometric and the second one is analytical.
There are functions for which the derivative at a point does not exist, consider $f : R \to R$ defined by $f(x) = |x|$. Then f is not differentiable at $x = 0$, note that there is a sharp edge at $x = 0$. As we have already seen that the existence of limit of a function f as $x \to x_0$ is meaningful and interesting only when x_0 is a limit point of the domain of the definition of the function, so without loss of generality we assume that the point under consideration is a limit point of the domain of the function.

Definition 0.3 (Differential of a function)**.** Let $f : S \subset \mathbb{R} \to \mathbb{R}$ and $x \in S$ be such that there exists $\delta > 0$ such that $(x - \delta, x + \delta) \subset S$. The derivative of the function f at x is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h},$$

if the limit exists. Let $y = f(x), \Delta x = h, \Delta y = f(x + h) - f(x)$. The quantity Δy denotes the change in the value of the dependent variable y corresponding to the change Δx of the independent variable x. Then $f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$. Consider the function $\epsilon : (-\delta, \delta) \to \mathbb{R}$ defined as

$$\epsilon(h) = \frac{f(x + h) - f(x)}{h} - f'(x) = \frac{\Delta y}{\Delta x} - f'(x).$$



This article is written by Prof. Kallol Paul, Department of Mathematics, Jadavpur University for UG I Math students 2024

It is easy to see that the existence of the derivative of the function f at x is equivalent to the fact that $\lim_{h\to 0}\epsilon(h)=0$. Now,

$$\Delta y = f'(x)\Delta x + \epsilon(\Delta x)\Delta x$$

indicates that the change Δy is sum of two parts, one is linear part $f'(x)\Delta x$ and the other part is $\epsilon(\Delta x)\Delta x$, which can be made as small as possible compared to Δx by making Δx itself small enough. The linear part in the expression Δy is known as the differential of the function $y=f(x)$ and we write it as

$$dy = df(x) = f'(x)\Delta x.$$

Thus for a function f which is differentiable at x, the differential of f is a well-defined function of Δx. For the particular function $y=x$, we get $dx=\Delta x$. Thus the differential of the function $y=f(x)$ can be written as

$$dy = df(x) = f'(x)dx.$$

Example 0.4. Let $f:\mathbb{R}\to\mathbb{R}$ be defined as $f(x)=x^2$. Consider the point $x=2$. Clearly

$$\Delta y = (2+\Delta x)^2 - 2^2 = 4.\Delta x + \Delta x^2$$

and so the differential of f at $x=2$ is $dy = df(x)|_{x=2} = 4dx$.

Theorem 0.5. *Let $f:S\subset R\to R$ and $x_0\in S$ be a limit point of S. Then f is differentiable at x_0 if and only if there exists a unique function $\phi:S\to R$ such that ϕ is continuous at x_0 and*

$$f(x)-f(x_0) = (x-x_0)\phi(x)\ \forall x\in S.$$

Moreover, $\phi(x_0)=f'(x_0)$.

Proof. We first prove the necessary part. Define $\phi:S\to R$ by

$$\begin{aligned}\phi(x) &= \frac{f(x)-f(x_0)}{x-x_0},\ x\neq x_0\\ &= f'(x_0),\quad \text{otherwise.}\end{aligned}$$

Then $\lim_{x\to x_0}\phi(x) = f'(x_0) = \phi(x_0)$ and so ϕ is continuous at x_0. Also from the definition of ϕ it follows that

$$f(x)-f(x_0) = (x-x_0)\phi(x)\ \forall x\in S.$$

Next, we show that such function ϕ is unique. Let there be another function $\phi_1:S\to R$ such that ϕ_1 is continuous at x_0 and

$$f(x)-f(x_0) = (x-x_0)\phi_1(x)\ \forall x\in S.$$

Then for each $x(\neq x_0)\in S$, $\phi(x)=\phi_1(x)$. Since ϕ,ϕ_1 are continuous at x_0, it follows that $\phi(x_0)=\phi_1(x_0)$. Thus $\phi(x)=\phi_1(x)$ for all $x\in S$ and so $\phi=\phi_1$. This completes the proof of necessary part.

We next prove the sufficient part. Suppose $\phi:S\to R$ is continuous at x_0 and

$$f(x)-f(x_0) = (x-x_0)\phi(x)\ \forall x\in S.$$

Then for all $x\neq x_0$ we get $\phi(x)=\frac{f(x)-f(x_0)}{x-x_0}$ and by continuity of ϕ at x_0 we conclude that

$$\phi(x_0) = \lim_{x\to x_0}\phi(x) = \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}.$$

Thus f is differentiable at x_0.

□

Theorem 0.6. *Let $f : S \subset R \to R$ and $x_0 \in S$ be such that there exists $\delta > 0$ such that $(x_0-\delta, x_0+\delta) \subset S$. Then f is differentiable at x_0 if and only if there exists a unique function $\phi : (-\delta, \delta) \to R$ such that ϕ is continuous at 0 and*

$$f(x_0 + h) - f(x_0) = h\phi(h)\ \forall h \in (-\delta, \delta).$$

Moreover, $\phi(0) = f'(x_0)$.

Proof. We first prove the necessary part. Define $\phi : (-\delta, \delta) \to R$ by

$$\begin{aligned}\phi(h) &= \frac{f(x_0 + h) - f(x_0)}{h},\ h \neq 0 \\ &= f'(x_0),\quad \text{otherwise.}\end{aligned}$$

Then $\lim_{h\to 0} \phi(h) = f'(x_0) = \phi(0)$ and so ϕ is continuous at 0. Also from the definition of ϕ it follows that

$$f(x_0 + h) - f(x_0) = h\phi(h)\ \forall h \in (-\delta, \delta).$$

Next, we show that such function ϕ is unique. Let there be another function $\phi_1 : (-\delta, \delta) \to R$ such that ϕ_1 is continuous at 0 and

$$f(x_0 + h) - f(x_0) = h\phi_1(h)\ \forall h \in (-\delta, \delta).$$

Then for each $h(\neq 0) \in (-\delta, \delta)$, $\phi(h) = \phi_1(h)$. Since ϕ, ϕ_1 are continuous at 0, it follows that $\phi(0) = \phi_1(0)$. Thus $\phi(h) = \phi_1(h)$ for all $h \in (-\delta, \delta)$ and so $\phi = \phi_1$. This completes the proof of necessary part.

We next prove the sufficient part. Suppose $\phi : (-\delta, \delta) \to R$ is continuous at 0 and

$$f(x_0 + h) - f(x_0) = h\phi(h)\ \forall h \in (-\delta, \delta).$$

Then for all $h \neq 0$ we get $\phi(h) = \frac{f(x_0+h)-f(x_0)}{h}$ and by continuity of ϕ at 0 we conclude that

$$\phi(0) = \lim_{h\to 0} \phi(h) = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

Thus f is differentiable at x_0. □

Theorem 0.7 (Differentiability implies continuity)**.** *Let $S \subset \mathbb{R}$ and x_0 be a limit point of S. If f is differentiable at x_0 then f is continuous at x_0 but the converse is not true.*

Proof. Since f is differentiable at x_0 so there exists a function ϕ defined on S, which is continuous at x_0 and satisfies

$$f(x) - f(x_0) = (x - x_0)\phi(x)\ \forall x \in S.$$

Taking limit as $x \to x_0$ we get

$$\lim_{x\to x_0} f(x) = f(x_0).$$

This shows that f is continuous at x_0. For the converse part, consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$.

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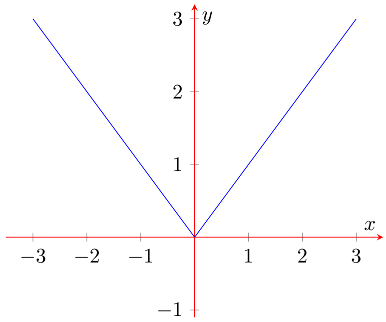

Then f is continuous at 0. Observe that

$$f'(0+) = \lim_{x\to 0+} \frac{|x|}{x} = \lim_{x\to 0+} \frac{x}{x} = \lim_{x\to 0+} 1 = 1,$$

$$\text{whereas, } f'(0-) = \lim_{x\to 0-} \frac{|x|}{x} = \lim_{x\to 0-} \frac{-x}{x} = \lim_{x\to 0+} -1 = -1.$$

So the function f is not differentiable at 0. □

Theorem 0.8 (Chain rule)**.** *Let $S, T \subset \mathbb{R}$ and $x_0 \in S$ be a limit point of S. Also assume $y_0 \in T$ is a limit point of T. Let $f : S \to T$ and $g : T \to \mathbb{R}$ be two functions such that f is differentiable at x_0 and g is differentiable at $f(x_0) = y_0$. Then $g \circ f : S \to \mathbb{R}$ is differentiable at x_0 and*

$$(g \circ f)'(x_0) = g'(y_0)f'(x_0).$$

Proof. Since f is differentiable at x_0 so there exists a function ϕ defined on S, which is continuous at x_0 and satisfies

$$f(x) - f(x_0) = (x - x_0)\phi(x) \ \forall x \in S.$$

Again, Since g is differentiable at y_0 so there exists a function ψ defined on T, which is continuous at y_0 and satisfies

$$g(y) - g(y_0) = (y - y_0)\psi(y) \ \forall y \in T.$$

Also observe that, $\phi(x_0) = f'(x_0)$ and $\psi(y_0) = \psi(f(x_0)) = g'(y_0)$. Now,

$$\begin{aligned} g \circ f(x) - g \circ f(x_0) &= g(f(x)) - g(f(x_0)) \\ &= \psi(f(x))(f(x) - f(x_0)) \\ &= \psi \circ f(x).\phi(x)(x - x_0). \end{aligned}$$

Since composition of two continuous function is continuous so $\psi \circ f$ is continuous at x_0, also product of two continuous function is continuous so that $(\psi \circ f).\,\phi$ is continuous at x_0. Defining $h : S \to \mathbb{R}$ by $h(x) = \psi \circ f(x).\phi(x)$ we see that h is continuous at x_0 and h satisfies

$$g \circ f(x) - g \circ f(x_0) = h(x)(x - x_0) \ \forall x \in S.$$

This shows that $g \circ f$ is differentiable at x_0 and

$$(g \circ f)'(x_0) = g'(y_0)f'(x_0).$$

□

Remark 0.9. To illustrate the above result let us look at the function $u(x) = \cos(x^3 + 3x)$. Consider $f(x) = x^3+3x$ and $g(x) = \cos x$. Then $g\circ f(x) = g(f(x)) = \cos(x^3+3x)$. Thus $u'(x) = g'(f(x))f'(x) = -\sin(x^3 + 3x).(3x^2 + 3) = -3(x^2 + 1)\sin(x^3 + 3x)$.

Theorem 0.10. *Let $X, Y \subset \mathbb{R}$ and $f : X \to Y$ be a function which is invertible. Let $x_0 \in X$ and $f(x_0) = y_0$. If f is differentiable at x_0 and f^{-1} is differentiable at y_0 then*

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}.$$

Proof. Clearly $(f^{-1} \circ f)'(x_0) = (f^{-1})'(y_0).f'(x_0)$ and $f^{-1} \circ f = I_X$, where I_X is the identity function on X. Then we get, $(f^{-1})'(y_0).f'(x_0) = 1$ and so $(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$. □

The next theorem shows that differentiability of f^{-1} at y_0 is not essential in the hypothesis.

Theorem 0.11. *Let $X, Y \subset \mathbb{R}$ and $f : X \to Y$ be a function which is invertible. Let $x_0 \in X$ and $f(x_0) = y_0$. If f is differentiable at x_0 and f^{-1} is continuous at y_0, $f'(x_0) \neq 0$ then f^{-1} is differentiable at y_0 and*

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}.$$

Proof. Since f is differentiable at x_0 so there exists a unique function ϕ defined on X, which is continuous at x_0 and satisfies

$$f(x) - f(x_0) = (x - x_0)\phi(x) \ \forall x \in X.$$

Also $\phi(x_0) = f'(x_0) \neq 0$. Then by the neighbourhood property of the continuous function ϕ there exists $\delta > 0$ such that $\phi(x) \neq 0$ for all $x \in (x_0 - \delta, x_0 + \delta) \cap X$. Since f^{-1} is continuous at y_0 so $(f^{-1})^{-1}((x_0 - \delta, x_0 + \delta) \cap X)$ is an open set in Y containing the point y_0. Let $U = (f^{-1})^{-1}((x_0 - \delta, x_0 + \delta) \cap X)$. Then for all $y \in U$ we get,

$$y - y_0 = f \circ f^{-1}(y) - f \circ f^{-1}(y_0) = f(f^{-1}(y)) - f(f^{-1}(y_0)) = \phi(f^{-1}(y))(f^{-1}(y) - f^{-1}(y_0)).$$

Observe that $\phi \circ f^{-1}(y) = \phi(f^{-1}(y)) \neq 0$ for all $y \in U$ and so the function $\frac{1}{\phi\circ f^{-1}}$ is continuous at y_0. This along with the relation

$$f^{-1}(y) - f^{-1}(y_0) = \frac{1}{\phi \circ f^{-1}(y)}(y - y_0)$$

implies that f^{-1} is differentiable at y_0 and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}.$$

□

Remark 0.12. The condition $f'(x_0) \neq 0$ is essential. Consider $f : [0, 1] \to [0, 1]$ defined as $f(x) = x^3$. Then f is differentiable at 0 and $f'(0) = 0$. Clearly f is bijective and so f^{-1} exists. But $f^{-1}(x) = x^{1/3}$ is not differentiable at 0, though continuous at 0.

Algebra of differentiable functions.

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Theorem 0.13. *Let $S \subset \mathbb{R}$ and x_0 be a limit point of S. Let $f, g : S \to \mathbb{R}$ be differentiable at x_0. Then*
(i) $f + g$ is differentiable at x_0 and $(f + g)'(x_0) = f'(x_0) + g'(x_0)$.
(ii) $f.g$ is differentiable at x_0 and

$$(f.g)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0).$$

(iii) For any real constant c, cf is differentiable at x_0 and $(cf)'(x_0) = cf'(x_0)$.
(iv) if $g(x) \neq 0$ for all $x \in S$ then $\frac{1}{g}$ is differentiable at x_0 and $(\frac{1}{g})'(x_0) = -\frac{g'(x_0)}{g(x_0)^2}$.
(v) if $g(x) \neq 0$ for all $x \in S$ then $\frac{f}{g}$ is differentiable at x_0 and

$$\left(\frac{f}{g}\right)'(x_0) = -\frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}.$$

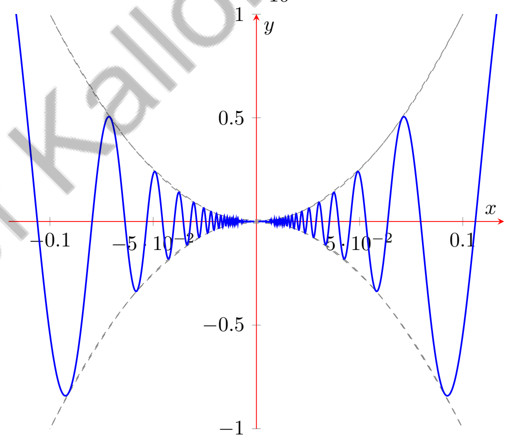
Remark 0.14. It may so happen that f is not differentiable at a point x_0 but g is differentiable at x_0 and the product $f.g$ is differentiable at x_0. Consider $f, g : \mathbb{R} \to \mathbb{R}$ defined as

$$\begin{aligned} f(x) &= \sin(1/x), \; x \neq 0 \\ &= 0, \; x = 0 \\ g(x) &= x^2. \end{aligned}$$

Then $f.g : \mathbb{R} \to \mathbb{R}$ is given by

$$f.g(x) = x^2 \sin(1/x),$$

which is differentiable at 0, though f is not differentiable at 0.



Definition 0.15 (Extreme value of a function). Let $f : S \subset \mathbb{R} \to \mathbb{R}$. Then
(i) f is said to have a relative maximum or maximum at a point $x_0 \in S$ if there exists $\delta > 0$ such that

$$f(x) \leq f(x_0, \; \forall x \in (x_0 - \delta, x_0 + \delta) \cap S.$$

(ii) f is said to have a relative minimum or minimum at a point $x_0 \in S$ if there exists $\delta > 0$ such that

$$f(x) \geq f(x_0), \; \forall x \in (x_0 - \delta, x_0 + \delta) \cap S.$$

(iii) f is said to have a global maximum at a point $x_0 \in S$ if

$$f(x) \leq f(x_0),\ \forall x \in S.$$

(iv) f is said to have a global minimum at a point $x_0 \in S$ if

$$f(x) \geq f(x_0),\ \forall x \in S.$$

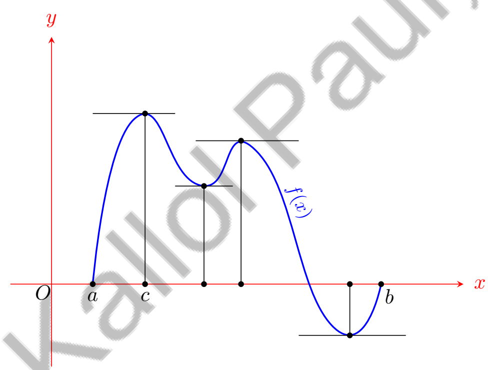
The function f is said to have a relative extremum at a point x_0 if the function has either relative maximum or relative minimum at x_0.

Theorem 0.16. *Let $f : S \subset \mathbb{R} \to \mathbb{R}$ be such that f attains its relative maxima (or minima) at an interior point $c \in S$. If f is differentiable at c then $f'(c) = 0$.*

Proof. Assume that f attains its maxima at $c \in S$. Then there exists $\delta > 0$ such that

$$f(x) \leq f(c),\ \forall x \in (c - \delta, c + \delta) \cap S.$$

Now, $f(c + h) - f(c) \leq 0,\ \forall h \in (-\delta, \delta)$ and so $f'(c+) = \lim_{h\to 0+} \frac{f(c+h)-f(c)}{h} \leq 0$, whereas $f'(c-) = \lim_{h\to 0-} \frac{f(c+h)-f(c)}{h} \geq 0$.

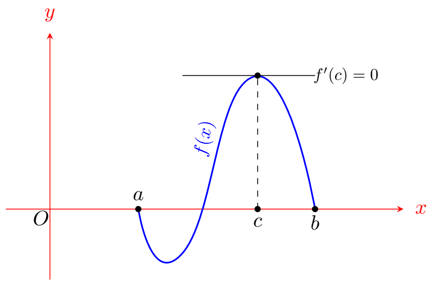


Since f is differentiable at c so the only possibility is that $f'(c+) = f'(c-) = 0$. Thus $f'(c) = 0$. The proof for the case when f attains its relative minima follows in the same spirit. □

Remark 0.17. Observe that the existence of derivative at a point is not necessary for a function to have maxima or minima at that point. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = |x|$. Then f has a minima at 0 but f is not differentiable at 0.

Theorem 0.18 (Rolle's Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be a function such that*
(i) f is continuous on $[a, b]$,
(ii) f' exists on (a, b) and
(iii) $f(a) = f(b) = 0$.
Then there exists a point $c \in (a, b)$ such that $f'(c) = 0$.

Proof. If $f(x) = 0$ for all $x \in [a, b]$ then clearly $f'(x) = 0$ for all $x \in [a, b]$ and the result holds. Without loss of generality assume that $f(x) > 0$ for some $x \in (a, b)$. Since f is continuous on the compact set $[a, b]$, f attains its global maximum at some point $c \in [a, b]$. As $f(a) = f(b) = 0$, so $c \in (a, b)$. Now, f attains its maxima at c and f is differentiable at c imply that $f'(c) = 0$.



□

Remark 0.19. The condition $f(a) = f(b)$ is sufficient to guarantee the existence of $c \in (a, b)$ such that $f'(c) = 0$. Since f is continuous on the closed interval $[a, b]$ so f attains its maximum and minimum therein. First suppose that f attains its extremum values at a and b. Then $f(a) = f(b)$ shows that f is constant on $[a, b]$ and so $f'(x) = 0$ for all $x \in [a, b]$. Next, assume f attains its maximum or minimum at some point $c \in (a, b)$. Then c, being interior point of $[a, b]$, we get $f'(c) = 0$.

Remark 0.20. The geometrical interpretation of Rolle's theorem is that there exists a point at which the tangent to the curve is parallel to the x-axis.

The following examples illustrate the fact that all the three conditions mentioned in Theorem 0.18 are needed.

Example 0.21. (i) Let $f : [0, 1] \to \mathbb{R}$ be defined as

$$\begin{aligned} f(x) &= \frac{1}{x}, \ x \in (0, 1) \\ &= 0, \ x = 0, 1 \end{aligned}$$

Then f satisfies conditions (ii), (iii) but does not satisfy (i). Note that there does not exist $c \in (0, 1)$ such that $f'(c) = 0$.
(ii) Let $f : [-1, 1] \to \mathbb{R}$ be defined as

$$f(x) = |x|.$$

Then f satisfies conditions (i), (iii) but does not satisfy (ii). Note that there does not exist $c \in (-1, 1)$ such that $f'(c) = 0$.
(iii) Let $f : [1, 2] \to \mathbb{R}$ be defined as

$$f(x) = x.$$

Then f satisfies conditions (i), (ii) but does not satisfy (iii). Note that there does not exist $c \in (1, 2)$ such that $f'(c) = 0$.

Theorem 0.22 (Mean Value theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be a function such that*
(i) f is continuous on $[a, b]$ and
(ii) f' exists on (a, b).
Then there exists $c \in (a, b)$ such that

$$f(b) - f(a) = (b - a)f'(c).$$

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Proof. Consider the function $\phi : [a, b] \to \mathbb{R}$ defined as

$$\phi(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

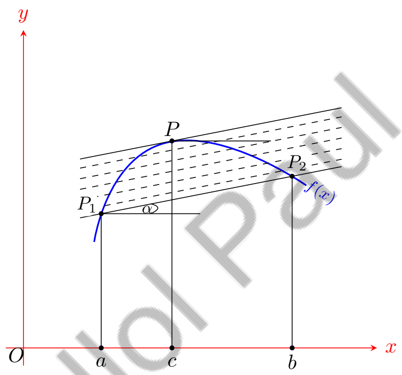
Then clearly ϕ is continuous on $[a, b]$, ϕ' exists on (a, b) and $\phi(a) = \phi(b) = 0$. So by Rolle's theorem there exists $c \in (a, b)$ such that $\phi'(c) = 0$. Observe that

$$\phi'(x) = f'(x) - \frac{f(b) - f(a)}{b - a} \ \forall x \in (a, b).$$

Thus we get

$$f(b) - f(a) = (b - a)f'(c) \ \text{ for some } c \in (a, b).$$

□



Remark 0.23. (i) This theorem is known as Lagrange's Mean Value Theorem or Mean Value Theorem of differential Calculus.
(ii) This theorem can also be stated as :
(a) Let $f : [x_1, x_2] \to \mathbb{R}$ be a function such that f is continuous on $[x_1, x_2]$ and f' exists on (x_1, x_2). Then there exists $\theta \in (0, 1)$ such that

$$f(x_2) - f(x_1) = (x_2 - x_1)f'(x_1 + (x_2 - x_1)\theta).$$

Note that each real number in $[x_1, x_2]$ can be written in the form $x_1 + (x_2 - x_1)\theta$, for $\theta \in [0, 1]$.
(b) Let $f : [x, x + h] \to \mathbb{R}$ be a function such that f is continuous on $[x, x + h]$ and f' exists on $(x, x + h)$. Then there exists $\theta \in (0, 1)$ such that

$$f(x + h) - f(x) = hf'(x + \theta h).$$

(iii) The geometrical interpretation of Lagrange's mean value theorem is that there is a point $c \in (a, b)$ at which the tangent to the curve is parallel to the chord joining the end points $(a, f(a))$ and $(b, f(b))$.

Theorem 0.24 (Cauchy Mean Value theorem)**.** *If f and g are continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) then there exists a point $c \in (a, b)$ such that*

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c).$$

If $g'(x) \neq 0$ for all $x \in (a, b)$, then

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Proof. Consider the function h defined on $[a, b]$ as

$$h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x).$$

Then h is continuous on closed interval $[a, b]$ and differentiable on open interval (a, b). So there exists $c \in (a, b)$ such that

$$h'(c) = \frac{h(b) - h(a)}{b - a}.$$

Clearly $h(a) = f(b)g(a) - g(b)f(a) = h(b)$ so that

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c).$$

□

Remark 0.25. The geometrical interpretation of Cauchy's mean value theorem is that there is a point $c \in (a, b)$ at which the tangent to the curve $h(x) = (f(x), g(x))$ is parallel to the secant joining the end points $(f(a), g(a))$ and $(f(b), g(b))$.

Theorem 0.26 (L'Hospital's rule ($\frac{0}{0}$form))**.** *Suppose f and g are real valued functions differentiable on (a, b) and $g'(x) \neq 0$ for all $x \in (a, b)$. Suppose*

$$\frac{f'(x)}{g'(x)} \to K \text{ as } x \to a.$$

If $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$ then

$$\frac{f(x)}{g(x)} \to K \text{ as as } x \to a.$$

Proof. Consider first $-\infty < K < \infty$ and $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$. Let $\epsilon > 0$. From the definition of limit it follows that there exists $c \in (a, b)$ such that for all $x, a < x < c$ we have

$$K - \epsilon < \frac{f'(x)}{g'(x)} < K + \epsilon.$$

If $a < x < y < c$ then by Cauchy's mean value theorem there exists $t \in (x, y)$ such that

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(t)}{g'(t)}.$$

Thus for all x, y with $a < x < y < c$, we get

$$K - \epsilon < \frac{f(x) - f(y)}{g(x) - g(y)} < K + \epsilon.$$

Letting $x \to a$ we get

$$K - \epsilon \leq \frac{f(y)}{g(y)} \leq K + \epsilon, \ \forall y \in (a, c).$$

Next consider $K = \infty$ and $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$. Then given $M > 0$ there exists $c \in (a, b)$ such that

$$\frac{f'(x)}{g'(x)} > M, \ \forall x \in (a, c).$$

Then as above we get for all x, y with $a < x < y < c$,

$$\frac{f(x) - f(y)}{g(x) - g(y)} > M.$$

Letting $x \to a$ we get for all $y, y \in (a, c)$,

$$\frac{f(y)}{g(y)} \geq M.$$

This shows that

$$\frac{f(x)}{g(x)} \to \infty \text{ as } x \to a.$$

The case for $K = -\infty$ can be dealt similarly. □

Theorem 0.27 (L'Hospital's rule ($\frac{1}{\infty}$ form)). *Suppose f and g are real valued functions differentiable on (a, b) and $g'(x) \neq 0$ for all $x \in (a, b)$. Suppose*

$$\frac{f'(x)}{g'(x)} \to K \text{ as } x \to a.$$

If $g(x) \to \infty$ as $x \to a$, then

$$\frac{f(x)}{g(x)} \to K \text{ as as } x \to a.$$

Proof. First consider the case $-\infty < K < \infty$ and $g(x) \to \infty$ as $x \to a$. Let $\epsilon > 0$. From the definition of limit it follows that there exists $c \in (a, b)$ such that for all $x, a < x < c$ we have

$$K - \epsilon < \frac{f'(x)}{g'(x)} < K + \epsilon.$$

If $a < x < y < c$ then by Cauchy's mean value theorem there exists $t \in (x, y)$ such that

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(t)}{g'(t)}.$$

Thus for all x, y with $a < x < y < c$, we get

$$K - \epsilon < \frac{f(x) - f(y)}{g(x) - g(y)} < K + \epsilon.$$

Keeping y fixed we can choose a point $c_1 \in (a, y)$ such that $g(x) > g(y)$ and $g(x) > 0$ if $a < x < c_1$. Then

$$(K - \epsilon)\Big(1 - \frac{g(y)}{g(x)}\Big) < \frac{g(x) - g(y)}{g(x)}.\frac{f(x) - f(y)}{g(x) - g(y)} < (K + \epsilon)\Big(1 - \frac{g(y)}{g(x)}\Big).$$

Thus,

$$(K - \epsilon)\Big(1 - \frac{g(y)}{g(x)}\Big) < \frac{f(x)}{g(x)} - \frac{f(y)}{g(x)} < (K + \epsilon)\Big(1 - \frac{g(y)}{g(x)}\Big).$$

Letting $x \to a$ and noting that $g(x) \to \infty$ as $x \to a$ we get,

$$(K - \epsilon) \leq \lim_{x \to a} \frac{f(x)}{g(x)} \leq (K + \epsilon).$$

Since $\epsilon > 0$ is arbitrary we get

$$\lim_{x \to a} \frac{f(x)}{g(x)} = K.$$

Next consider the case $K = \infty$. Let $\alpha > 1$. Choose $c \in (a, b)$ such that $\frac{f'(x)}{g'(x)} > \alpha$ for all $x \in (a, c)$. Then for all $a < x < y \leq c$, using Cauchy's mean value theorem, we get

$$\frac{f(x) - f(y)}{g(x) - g(y)} > \alpha. \tag{1}$$

Since $g(x) \to \infty$ as $x \to a$ so we can assume that $g(c) > 0$ and $\frac{|f(c)|}{g(x)} < \frac{1}{2}$, $0 < \frac{g(c)}{g(x)} < \frac{1}{2}$ for all $x \in (a, c)$. Choosing $y = c$ in the inequality (1) we get

$$\Big(1 - \frac{g(c)}{g(x)}\Big)\Big(\frac{f(x) - f(c)}{g(x) - g(c)}\Big) > \alpha\Big(1 - \frac{g(c)}{g(x)}\Big) > \frac{\alpha}{2}.$$

This implies that

$$\frac{f(x)}{g(x)} > \frac{1}{2}\alpha + \frac{f(c)}{g(x)} > \frac{1}{2}(\alpha - 1).$$

Since $\alpha > 1$ is arbitrary it follows that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = K = \infty.$$

The case for $K = -\infty$ can be dealt similarly. □

Example 0.28. Evaluate $\lim\limits_{x \to 0} \left(\frac{1}{x} - \frac{1}{x^2}\log(1 + x)\right)$.

Solution. $\lim\limits_{x \to 0} \left(\frac{1}{x} - \frac{1}{x^2}\log(1 + x)\right) = \lim\limits_{x \to 0} \left(\frac{x - \log(1+x)}{x^2}\right)$. Consider the functions $f : (0, 1) \to \mathbb{R}$ and $g : (0, 1) \to \mathbb{R}$ defined as $f(x) = x - \log(1 + x)$ and $g(x) = x^2$, respectively. Clearly, f, g are differentiable on $(0, 1)$ and $g'(x) = 2x \neq 0$ for all $x \in (0, 1)$. Now, $f(x) \to 0$ and $g(x) \to 0$ as $x \to 0$. Therefore, applying L'Hospital's rule we get,

$$\begin{aligned}
\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{x^2}\log(1 + x)\right) &= \lim_{x \to 0} \frac{f(x)}{g(x)} \\
&= \lim_{x \to 0} \frac{f'(x)}{g'(x)} \\
&= \lim_{x \to 0} \frac{1}{2(1 + x)} \\
&= \frac{1}{2}.
\end{aligned}$$

Example 0.29. Evaluate $\lim\limits_{x \to 0}(\cot x)^{\sin x}$.

Solution. Let $l = \lim\limits_{x \to 0}(\cot x)^{\sin x}$. Then $\log l = \lim\limits_{x \to 0} \frac{\log(\cot x)}{\text{cosec } x}$. Consider the functions $f : (0, 1) \to \mathbb{R}$ and $g : (0, 1) \to \mathbb{R}$ defined as $f(x) = \log(\cot x)$ and $g(x) = \text{cosec } x$, respectively. Clearly, f, g are differentiable on $(0, 1)$ and $g'(x) = -\text{cosec } x \cot x \neq 0$ for all $x \in (0, 1)$. Now, $g(x) \to \infty$ as $x \to 0$ and so applying L'Hospital's rule we get,

$$\begin{aligned}
\log l &= \lim_{x \to 0} \frac{f(x)}{g(x)} \\
&= \lim_{x \to 0} \frac{f'(x)}{g'(x)} \\
&= \lim_{x \to 0} \frac{-(\frac{1}{\cot x})(\text{cosec}^2\, x)}{-\text{cosec } x \cot x} \\
&= \lim_{x \to 0} \sin x \sec^2 x = 0.
\end{aligned}$$

Therefore, $l = e^0 = 1$.

As applications of Lagrange's Mean Value Theorem, we can study the monotonicity of a function.

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Theorem 0.30. *Let f be a differentiable function defined on an interval $I \subset \mathbb{R}$. Then*
(i) $f'(x) \geq 0$ if and only if f is increasing on the interval I.
(ii) $f'(x) \leq 0$ if and only if f is decreasing on the interval I.
(iii) $f'(x) = 0$ for all $x \in I$ if and only if f is constant.

Proof. We prove (i), the proof of (ii) follows in the same spirit. Assume first that $f'(x) \geq 0$ for all $x \in I$. Let $x_1, x_2 \in I$ and $x_2 > x_1$. Then using the Mean Value theorem for the function f on the closed interval $[x_1, x_2]$ we get,

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1),$$

where $c \in (x_1, x_2)$. Now, $f'(c) \geq 0$ and $x_2 - x_1 > 0$ imply that $f(x_2) \geq f(x_1)$. Thus f is increasing on I. On the other hand suppose that f is increasing on I. We want to show that $f'(x) \geq 0$ for all $x \in I$. Observe that for small h, with $h \neq 0$

$$\frac{f(x+h) - f(x)}{h} \geq 0, \text{ if } h > 0$$
$$\frac{f(x) - f(x+h)}{-h} \geq 0, \text{ if } h < 0$$

Thus $f'(x+) \geq 0$ and $f'(x-) \geq 0$ which shows that $f'(x) \geq 0$.
The proof of (iii) follows easily from the Mean Value theorem. □

The non-negativity of differentiability at a point does not induce the monotonicity. For example consider the function$f : [-2\pi, 2\pi] \to \mathbb{R}$ defined by

$$\begin{aligned} f(x) = & \quad x + 4x^2 \sin\left(\tfrac{1}{x}\right), \quad x \neq 0 \\ = & \quad 0, \quad x = 0. \end{aligned}$$

Then f is differentiable on $[-2\pi, 2\pi]$ and

$$\begin{aligned} f'(x) = & \quad 1 + 8x \sin\left(\tfrac{1}{x}\right) - 4\cos\left(\tfrac{1}{x}\right), \quad x \neq 0 \\ = & \quad 1, \quad x = 0. \end{aligned}$$

Thus $f'(0) = 1 > 0$ but f is neither increasing nor decreasing in a neighbourhood of 0.

Motivation behind Taylor's formula

Consider a polynomial $p(x)$ in x of order n as

$$p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n.$$

Putting $x = a + h$ and expanding each term in powers of h we get

$$p(a+h) = c_0 + c_1 h + c_2 h^2 + \ldots + c_n h^n, \tag{2}$$

where $c_0, c_1, \ldots, c_n$ are constants independent of h. Putting $h = 0$ in equation (2) we get $c_0 = p(a)$. Differentiating (1) w.r.t. h and putting $h = 0$ we get $c_1 = p'(a)$. Proceeding in this way differentiating successively and putting $h = 0$ we get

$$c_k = \frac{1}{k!}\, p^k(a) \text{ for each } k = 0, 1, 2, \ldots, n.$$

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Thus the Taylor's formula for polynomials is

$$p(a+h) = p(a) + hp'(a) + \frac{h^2}{2!}p''(a) + \ldots + \frac{h^n}{n!}p^n(a).$$

This formula which holds for polynomials also holds for non-polynomial functions with some modifications provided they satisfy certain conditions. This was first observed by Taylor, a pupil of Newton.

Theorem 0.31 (Taylor's formula)**.** *Let $f : [a,b] \to \mathbb{R}$ be a function such that*
(i) $f, f', f'', \ldots, f^{n-1}$ are continuous on the closed interval $[a,b]$ and
(ii) $f, f', f'', \ldots, f^n$ exist on the open interval (a,b).
Assume that u, v are two distinct points in $[a,b]$. Define $p : [a,b] \to \mathbb{R}$ as

$$p(t) = f(u) + (t-u)f'(u) + \frac{(t-u)^2}{2!}f''(u) + \ldots + \frac{(t-u)^{n-1}}{(n-1)!}f^{n-1}(u).$$

Then there exists a point x between u and v such that

$$f(v) = p(v) + \frac{(v-u)^n}{(n)!}f^n(x)$$

$$\text{i.e., } f(v) = f(u) + (v-u)f'(u) + \frac{(v-u)^2}{2!}f''(u) + \ldots + \frac{(v-u)^{n-1}}{(n-1)!}f^{n-1}(u) + \frac{(v-u)^n}{n!}f^n(x).$$

The last term is known as the remainder term in Lagrange's form.

Proof. Consider the function $\phi : [a,b] \to \mathbb{R}$ defined as

$$\phi(t) = f(t) - p(t) - K(t-u)^n,$$

where the constant K is chosen in such a way that $\phi(u) = \phi(v)$. From the definition of $p(t)$ it follows that

$$p^k(u) = f^k(u) \ \forall k = 0, 1, 2, \ldots, n-1.$$

Observe that $\phi(u) = 0$ and so we can choose $K = \frac{f(v)-p(v)}{(v-u)^n}$. Also we get, $\phi(u) = \phi'(u) = \cdots = \phi^{n-1}(u) = 0$. The choice of K forces $\phi(v) = 0$ and so there exists $x_1 \in (u,v)$ such that $\phi'(x_1) = 0$. Since $\phi(u) = 0$ and $\phi'(x_1) = 0$ so there exists x_2 between u and x_1 such that $\phi''(x_2) = 0$. Proceeding in this way we get x_n between u and x_{n-1} (to be precise between u and v) such that $\phi^n(x_n) = 0$. Also we have for all $t \in (a,b)$

$$\phi^n(t) = f^n(t) - n!K.$$

This implies that $K = \frac{f^n(x_n)}{n!}$ and so naming $x_n = x$ we get

$$\frac{f(v)-p(v)}{(v-u)^n} - \frac{f^n(x_n)}{n!} = 0, \text{ i.e., } f(v) = p(v) + \frac{(v-u)^n}{(n)!}f^n(x).$$

Thus there exists x between u and v such that

$$f(v) = f(u) + (v-u)f'(u) + \frac{(v-u)^2}{2!}f''(u) + \ldots + \frac{(v-u)^{n-1}}{(n-1)!}f^{n-1}(u) + \frac{(v-u)^n}{n!}f^n(x).$$

□

Remark 0.32. This theorem can also be stated as Let $f : [a,b] \to \mathbb{R}$ be a function such that
(i) $f, f', f'', \ldots, f^{n-1}$ are continuous on the closed interval $[a,b]$ and

(ii) $f, f', f'', \ldots, f^n$ exist on the open interval (a, b).
Assume that $u, u+h$ are two distinct points in $[a, b]$. Define $p : [a, b] \to \mathbb{R}$ as

$$p(t) = f(u) + (t-u)f'(u) + \frac{(t-u)^2}{2!}f''(u) + \ldots + \frac{(t-u)^{n-1}}{(n-1)!}f^{n-1}(u).$$

Then there exists a point x between u and $u+h$ such that

$$f(u+h) = p(u+h) + \frac{h^n}{(n)!}f^n(x),$$

$$i.e., \ f(u+h) = f(u) + hf'(u) + \frac{h^2}{2!}f''(u) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(u) + \frac{h^n}{n!}f^n(x).$$

As an application of Taylor's formula we can find sufficient conditions for maxima or minima of a function f under certain conditions.

Theorem 0.33. *Let $f : [a, b] \to \mathbb{R}$ and c be an interior point of $[a, b]$. Assume that $f, f', f'', \ldots, f^n, f^n$ exist and are continuous on $(c-\delta, c+\delta)$ for some $\delta > 0$. Let $f'(c) = f''(c) = \ldots = f^{n-1}(c) = 0$ and $f^n(c) \neq 0$.*
(i) Then f has a maximum at c if n is even and $f^n(c) < 0$.
(ii) Then f has a minimum at c if n is even and $f^n(c) > 0$.
(iii) Then f has neither a maximum nor a minimum if n is odd.

Proof. Let $h \neq 0$ be such that $|h| < \delta$. Then using Taylor's formula we get,

$$f(c+h) = f(c) + hf'(c) + \frac{h^2}{2!}f''(c) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(c) + \frac{h^n}{n!}f^n(c+\theta h),$$

where $0 < \theta < 1$. Since $f^n(c) \neq 0$ and f^n is continuous at c so there exists $\delta_1 > 0$ such that $f^n(x) \neq 0$ for all $x \in (c-\delta_1, c+\delta_1)$. Choose $\delta_2 = \min\{\delta, \delta_1\}$. Then for all h with $|h| < \delta_2$ we have $f^n(c+\theta h) \neq 0$ and

$$f(c+h) - f(c) = \frac{h^n}{n!}f^n(c+\theta h).$$

We further observe that $f^n(c+\theta h)$ preserves the sign of $f^n(c)$.
(i) Assume that n is even and $f^n(c) < 0$. Then $h^n > 0$ and $f^n(c+\theta h) < 0$ so that

$$f(c+h) - f^(c) < 0, \ i.e., \ f(c+h) < f(c), \forall h \,\text{with}\, |h| < \delta.$$

Thus f has a maximum at c.
(ii) Assume that n is even and $f^n(c) > 0$. Then $h^n > 0$ and $f^n(c+\theta h) > 0$ so that

$$f(c+h) - f^(c) > 0, \ i.e., \ f(c+h) > f(c), \forall h \,\text{with}\, |h| < \delta.$$

Thus f has a minimum at c.
(iii) Assume that n is odd. First consider the case $f^n(c) > 0$. Then

$$\begin{aligned} f(c+h) - f(c) &= \tfrac{h^n}{n!}f^n(c+\theta h) \quad > 0, \text{ if } h > 0 \\ &= \tfrac{h^n}{n!}f^n(c+\theta h) \quad < 0, \text{ if } h < 0 \end{aligned}$$

Thus in a small neighbourhood of c $f(c+h) - f(c)$ is positive at some point and negative at some point which indicates that f neither has a maximum nor minimum at c. The case for $f^n(c) < 0$ can be dealt analogously. □

Theorem 0.34 (Darboux Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be a function such that f is differentiable at every point of $[a, b]$. If $f'(a) < \gamma < f'(b)$ then there exists $c \in (a, b)$ such that $f'(c) = \gamma$.*

Proof. Consider the function $\phi : [a, b] \to \mathbb{R}$ defined as $\phi(x) = f(x) - \gamma x$. Then ϕ is continuous on $[a, b]$ and ϕ is differentiable on $[a, b]$. Clearly $\phi'(x) = f'(x) - \gamma,\ \forall x \in [a, b]$. Observe that ϕ, being continuous on a compact set $[a, b]$, attains its minimum values at some point $c \in [a, b]$. Since ϕ is differentiable at c so we have $\phi'(c) = 0$, *i.e.*, $f'(c) = \gamma$. Observe $\phi'(a) = f'(a) - \gamma < 0$ so we get $c \neq a$. Also $c \neq b$. Thus $c \in (a, b)$ and $f'(c) = \gamma$. □

Remark 0.35. (i) Note that we did not assume the continuity of the function f' but f' satisfies the intermediate value property. The image of an interval under f' is an interval. This is inherent property of a differentiable function.
(ii) If f is differentiable on $[a, b]$ then f' can not have any simple discontinuities.

Example 0.36. Using Lagrange's mean value theorem, show that $\frac{z}{1+z} < \log(1 + z) < z,\ (z > 0)$.

Solution. Consider the function $f : [0, \infty) \to \mathbb{R}$ defined as $f(x) = \log(1 + x)$.
Then $f'(x) = \frac{1}{1+x}$. Let $z > 0$. Now f is continuous on $[0, z]$ and f' exists on $(0, z)$. So by Lagrange's mean value theorem, there exists $\theta \in (0, 1)$ such that $\frac{f(z)-f(0)}{z-0} = f'(\theta z)$ and so

$$\log(1 + z) = \frac{z}{1 + \theta z}.$$

As $\theta \in (0, 1)$ and $\theta z < z$, it follows that $1 < 1 + \theta z < 1 + z$. Hence $\frac{z}{1+z} < \frac{z}{1+\theta z} < z$. Thus

$$\frac{z}{1 + z} < \log(1 + z) < z \text{ for } z > 0.$$

Example 0.37. Show that $x < \sin^{-1} x < \frac{x}{\sqrt{1-x^2}},\ (0 < x < 1)$.

Solution. Consider the function $f : [0, 1) \to \mathbb{R}$ defined as $f(x) = \sin^{-1} x - x$.
Then $f'(x) = \frac{1}{\sqrt{1-x^2}} - 1 > 0$, for $x \in [0, 1)$. This implies that $f(x)$ is a strictly increasing function in $[0, 1)$ and so $f(x) > f(0)$ for $0 < x < 1$. As $f(0) = 0$, $f(x) > 0$ for $0 < x < 1$. Hence $\sin^{-1} x - x > 0$ for $0 < x < 1$ and so

$$x < \sin^{-1} x \text{ for } 0 < x < 1.$$

Next, consider the function $g : [0, 1) \to \mathbb{R}$ defined as $g(x) = \frac{x}{\sqrt{1-x^2}} - \sin^{-1} x$.
Then $g'(x) = \frac{\sqrt{1-x^2} + \frac{x(-2x)}{2\sqrt{1-x^2}}}{1-x^2} - \frac{1}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}}\left(\frac{1}{1-x^2} - 1\right) > 0$, for $x \in [0, 1)$. This implies that $f(x)$ is a strictly increasing function in $[0, 1)$ and so $f(x) > f(0)$ for $0 < x < 1$. As $f(0) = 0$, $f(x) > 0$ for $0 < x < 1$. Hence $\frac{x}{\sqrt{1-x^2}} - \sin^{-1} x > 0$ for $0 < x < 1$ and so

$$\sin^{-1} x < \frac{x}{\sqrt{1 - x^2}} \text{ for } 0 < x < 1.$$

Therefore, $x < \sin^{-1} x < \frac{x}{\sqrt{1-x^2}},\ (0 < x < 1)$.

Example 0.38. Using Taylor's formula, find the quadratic approximation of the function $f(x) = \sqrt{4 + x}$ at $x = 0$.

Solution. Consider the function $f : [-1, 1] \to \mathbb{R}$ defined as $f(x) = \sqrt{4 + x}$.
Then $f'(x) = \frac{1}{2\sqrt{4+x}} > 0,\ f''(x) = -\frac{1}{4(4+x)\sqrt{4+x}}$. Clearly f, f' are continuous on $[0, 1]$ and f, f', f'' are

exists on $(0,1)$. So by Taylor's formula the quadratic approximation of the function $f(x) = \sqrt{4+x}$ at $x = 0$ is

$$P(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2.$$

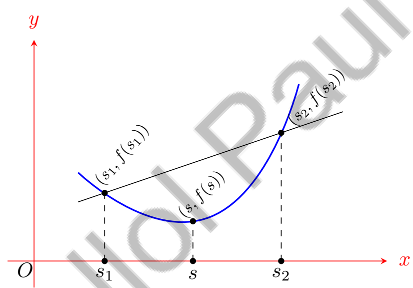
Now, $f(0) = 2, f'(0) = \frac{1}{4}, f''(0) = -\frac{1}{32}$. Therefore, the required quadratic polynomial is

$$P(x) = 2 + \frac{1}{4}x - \frac{1}{64}x^2.$$

Definition 0.39 (Convex function)**.** Let $S \subset \mathbb{R}$ be a convex set. Recall that a non-empty set S is said to be convex if $s_1, s_2 \in S$ and $t \in [0,1]$ implies that $(1-t)s_1 + ts_2 \in S$. A function $f : S \to \mathbb{R}$ is said to convex if for all $s_1, s_2 \in S$ and for all $t \in [0,1]$,

$$f((1-t)s_1 + ts_2) \leq (1-t)f(s_1) + tf(s_2).$$

Geometrically, it means that the functional value of the line segment joining s_1 and s_2 in the convex set S lies below the chord joining $(s_1, f(s_1))$ and $(s_2, f(s_2))$.



There is a nice connection between convexity and differentiability of a function on a convex set in terms of the second derivative. Observe that a convex function is not necessarily differentiable. Look at $f : [-1,1] \to \mathbb{R}$ defined by $f(x) = |x|$. Then f is convex but f is not differentiable at 0. On the other hand if we consider $f : [0, \pi/2] \to \mathbb{R}$ defined by $f(x) = \sin x$ then f is differentiable but f is not convex.

Theorem 0.40. *Let $f : (a,b) \to \mathbb{R}$ be twice differentiable on (a,b). Then f is convex on (a,b) if and only if $f''(x) \geq 0$ for all $x \in (a,b)$.*

Proof. Let $f : (a,b) \to \mathbb{R}$ be convex. Consider $x \in (a,b)$. Then

$$f''(x) = \lim_{h \to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}.$$

Consider h such that $x+h, x-h \in (a,b)$ and then by convexity of f we get

$$f(x) = f\Big(\frac{1}{2}(x+h) + \frac{1}{2}(x-h)\Big) \leq \frac{1}{2}f(x+h) + \frac{1}{2}f(x-h)$$

and this implies that

$$f(x+h) - 2f(x) + f(x-h) \geq 0.$$

Thus $f''(x) \geq 0$, $\forall x \in (a,b)$. □

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Conversely assume $f''(x) \geq 0$ for all $x \in (a,b)$. Let $u, v \in (a,b)$ and $t \in (0,1)$. Let $x_0 = (1-t)u + tv$. Then using Taylor's formula for the function f, there exists $x_1 \in (u, x_0)$ such that

$$f(u) = f(x_0) + (u - x_0)f'(x_0) + \frac{1}{2!}(u - x_0)^2 f''(x_1).$$

Similarly there exists $x_2 \in (x_0, v)$ such that

$$f(v) = f(x_0) + (v - x_0)f'(x_0) + \frac{1}{2!}(v - x_0)^2 f''(x_2).$$

So,

$$\begin{aligned}
(1-t)f(u) + tf(v) &= f(x_0) + 0.f'(x_0) + (1-t)\frac{1}{2!}(u - x_0)^2 f''(x_1) + t\frac{1}{2!}(v - x_0)^2 f''(x_2) \\
&= f(x_0) + M, \text{ (where } M \geq 0) \\
&\geq f(x_0) \\
&= f((1-t)u + tv).
\end{aligned}$$

Thus we get,

$$f((1-t)u + tv) \leq (1-t)f(u) + tf(v), \ \forall t \in [0,1]$$

This shows that f is convex on (a,b).

The next example of a function is the one which breaks our intuitive notion of continuity of a function. The example of such a function was first provided by Weierstrass. The geometric intuition does not hold anymore that for non-differentiability a sharp edge or vertex is there.

Example of a nowhere differentiable but everywhere continuous function.

Theorem 0.41. *There exists a function $f : \mathbb{R} \to \mathbb{R}$ such that f is continuous everywhere but differentiable nowhere.*

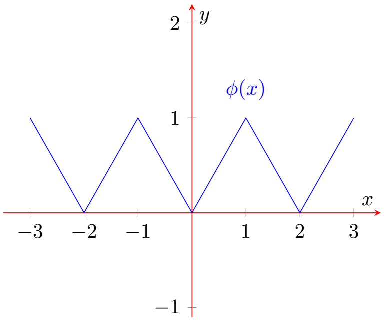
Proof. Define $\phi(x) = |x|$ for $-1 \leq x \leq 1$ and extend the definition of $\phi(x)$ to all real x by $\phi(x+2) = \phi(x)$. Then for all s, t such that

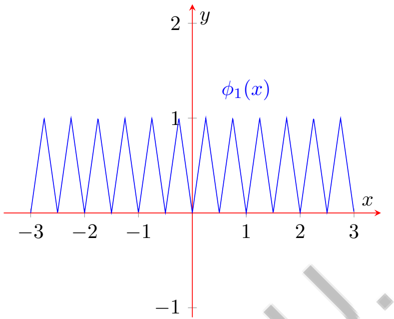
$$|\phi(s) - \phi(t)| \leq |s - t|.$$

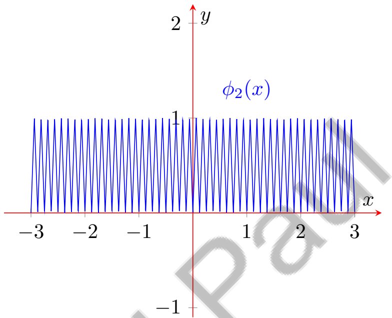
Clearly ϕ is continuous on $\mathbb{R}$. Define

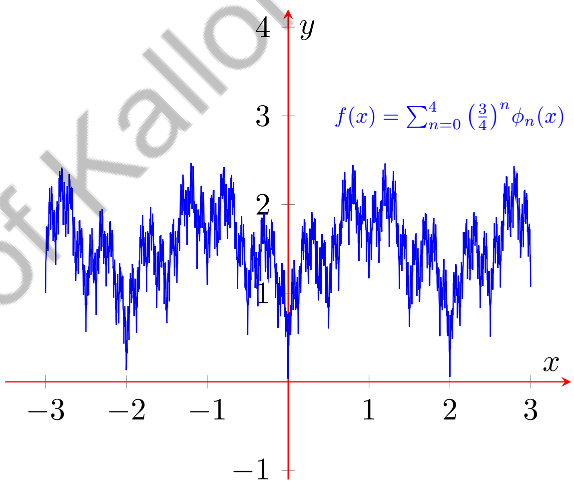
$$f(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \phi_n(x) \text{ where } \phi_n(x) = \phi(4^n x).$$

Since $0 \leq \phi \leq 1$.









By Weierstrass M-test the series $\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \phi_n(x)$ converges uniformly on $\mathbb{R}$. Since each ϕ_n is continuous so f is continuous.

Now fix a real number x and a positive integer m. Put

$$\delta_m = \pm\frac{1}{2}4^{-m},$$

where the sign is so chosen that no integer lies between $4^m x$ and $4^m(x+\delta_m)$. This can be done, since $4^m|\delta_m| = \frac{1}{2}$. Define

$$\gamma_n = \frac{\phi(4^n(x+\delta_m)) - \phi(4^n x)}{\delta_m},$$

when $n > m$ then $4^n\delta_m$ is an even integer, so that $\gamma_n = 0$ and when $0 \le n \le m$ $|\gamma_n| \le 4^n$. Since $|\gamma_m| = 4^m$, we conclude that

$$\begin{aligned}\Big|\frac{f(x+\delta_m) - f(x)}{\delta_m}\Big| &= \Big|\sum_{n=0}^{m}\Big(\frac{3}{4}\Big)^n \gamma_n\Big| \\ &\geq 3^m - \sum_{n=0}^{m-1} 3^n \\ &= \frac{1}{2}(3^m + 1).\end{aligned}$$

As $\delta_m \to 0$ so $_m \to \infty$. Hence f is not differentiable at x. □

Problems

(1) If $f(x) = |x|^3$, then compute $f'(x), f''(x)$ for all real x and show that $f^{(3)}(0)$ does not exist.

(2) Let $f : I \to \mathbb{R}$ be differentiable on an interval I. If f' is bounded on I then f satisfies Lipschitz condition on I.

(3) Suppose f is defined on an interval containing c and $f''(c)$ exists. Then show that

$$\lim_{h\to 0} \frac{f(c+h) - 2f(c) + f(c-h)}{h^2} = f''(c).$$

Give an example to show that the limit on the left hand side may exist, even if $f''(c)$ does not exist.

(4) Let f be defined for all real x, and suppose that

$$|f(x) - f(y)| \le (x-y)^2$$

for all real x and y. Prove that f is constant.

(5) Suppose f is defined in a neighbourhood of x and suppose $f''(x)$ exists. Show that

$$\lim_{h\to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x).$$

Show by an example that the limit may exist even if $f''(x)$ does not.

(6) If

$$C_0 + \frac{c_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{c_n}{n+1} = 0,$$

where $C_0, \ldots, C_n$ are real constants. Prove that the equation

$$C_0 + C_1 x + \cdots + C_{n-1}x^{n-1} + C_n x^n = 0$$

has at least one real root between 0 and 1.

(7) Suppose f is defined and differentiable for every $x > 0$ and $f'(x) \to 0$ as $x \to +\infty$. Put $g(x) = f(x+1) - f(x)$. Prove that $g(x) \to 0$ as $x \to +\infty$.

(8) Suppose

(a) f is continuous for $x \geq 0$,

(b) $f'(x)$ exists for $x > 0$,

(c) $f(0) = 0$,

(d) f' is monotonically increasing.

Put

$$g(x) = \frac{f(x)}{x} \ (x > 0)$$

and prove that g is monotonically increasing.

(9) Suppose $f'(x),\ g'(x)$ exist, $g'(x) \neq 0$ and $f(x) = g(x) = 0$. Prove that

$$\lim_{t \to x} \frac{f(t)}{g(t)} = \frac{f'(x)}{g'(x)}.$$

(10) Suppose $f'(x) > 0$ in (a, b). Prove that f is strictly increasing in (a, b) and let g be its inverse function. Prove that g is differentiable, and that

$$g'(f(x)) = \frac{1}{f'(x)} \ (a < x < b).$$

(11) Suppose f is differentiable in (a, b), $(a < x < b)$, $(x < \alpha_n < \beta_n)$ for $n = 1, 2, 3, \ldots$ and $\alpha_n \to x,\ \beta_n \to x$. Show that the quotients

$$\frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n}$$

need not converge to $f'(x)$, as $n \to \infty$, but that they do if we impose the additional assumption that the sequence $\{\frac{(\beta_n - x)}{(\beta_n - \alpha_n)}\}$ is bounded.

(12) Using Lagrange's mean value theorem, show that $\frac{x}{1+x^2} < \tan^{-1} x < x,\ (> 0)$.

(13) Use Lagrange's mean value theorem to prove that $|\sin x - \sin y| \leq |x - y|$ for all $x, y \in \mathbb{R}$.

(14) Show that $x + \frac{x^2}{2} < \log(1 + x) < x - \frac{x^2}{2(1+x)},\ (x > 0)$.

(15) Show that $\frac{x^2}{2(1+x)} < x - \log(1 + x) < \frac{x^2}{2},\ (x > 0)$.

(16) Show that $\sin x$ lies between $x - \frac{x^3}{6}$ and $x - \frac{x^3}{6} + \frac{x^5}{120}$.

(17) What is the third degree polynomial approximation of the function $f(x) = e^{4x}$ at $x = 0$.

(18) Assuming the validity of expansion, show that $\sin(e^x - 1) = x + \frac{x^2}{2!} + \frac{5x^4}{4!}$.

(19) Suppose f and g are complex differentiable functions on $(0, 1)$ and $f(x) \to 0, g(x) \to 0, f'(x) \to A, g'(x) \to B$ as $x \to 0$, where A and B are complex numbers, $B \neq 0$. Prove that

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \frac{A}{B}.$$

(20) Suppose g is a real function on $\mathbb{R}^1$, with bounded derivative (Say $|g'| \leq M$). Fix $\epsilon > 0$ and define $f(x) = x + \epsilon g(x)$. Prove that f is one-to-one if ϵ is small enough.

(21) Suppose f is differentiable function on (a, b). Then show that f is convex if f' is monotonically increasing.

(22) Let E be a closed subset of $\mathbb{R}$.

(i) Show that there exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ for which $E = \{x \in \mathbb{R} : f(x) = 0\}$.

(ii) Is it possible to find a function f which is differentiable on $\mathbb{R}$ such that $E = \{x \in \mathbb{R} : f(x) = 0\}$.

(iii) Can we find such a f which is n-times differentiable or is differentiable of all order?

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(23) Suppose f is differentiable on $[a, b]$ and $f(a) = 0$. Assume that there exists a constant K such that

$$|f'(x)| \leq K|f(x)|.$$

Prove that $f(x) = 0$ for all $x \in [a, b]$.

(24) Let f, g be differentiable functions on $\mathbb{R}$ and suppose that $f(0) = g(0)$. If $f'(x) \leq g'(x)$ for all $x \geq 0$ then prove that $f(x) \leq g(x)$ for all $x \geq 0$.

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§4 Taylor Theorems

§4.1 Some one-dimensional Taylor theorems

Remark. By *definition*, a function $f : \mathbb{R} \to \mathbb{R}$ which is continuous at 0 looks like a constant function near 0, in the sense that

$$f(t) = f(0) + \epsilon(t),$$

where $\epsilon(t) \to 0$ as $t \to 0$. By *definition*, again, a function $f : \mathbb{R} \to \mathbb{R}$ which is differentiable at 0 looks like a linear function near 0, in the sense that

$$f(t) = f(0) + f'(0)t + \epsilon(t)|t|,$$

where $\epsilon(t) \to 0$ as $t \to 0$. *Taylor's theorem* establishes that a function $f : \mathbb{R} \to \mathbb{R}$, if it is $(n-1)$ times differentiable in a neighbourhood of 0 and n times differentiable at 0, looks like a polynomial of degree n near 0, in the sense that

$$f(t) = f(0) + f'(0)t + \frac{f''(0)}{2!}t^2 + \cdots + \frac{f^{(n)}(0)}{n!}t^n + \epsilon(t)|t|^n,$$

where $\epsilon(t) \to 0$ as $t \to 0$.

Theorem 4.1 (A global Taylor's theorem)
If $f : (-a, a) \to \mathbb{R}$ is n times differentiable with $|f^{(n)}(t)| \leq M$ for all $t \in (-a, a)$, then

$$\left| f(t) - \sum_{j=0}^{n} \frac{f^{(j)}(0)}{j!}t^j \right| \leq \frac{M|t|^n}{n!}.$$

Theorem 4.2 (The local Taylor's theorem)
If $f : (-a, a) \to \mathbb{R}$, where $a > 0$ is $(n-1)$ times differentiable and $f^{(n)}(0)$ exists, then

$$f(t) = \sum_{j=0}^{n} \frac{f^{(j)}(0)}{j!}t^j + \epsilon(t)|t|^n,$$

where $\epsilon(t) \to 0$ then as $t \to 0$.

§5 Advanced Calculus in One Real Variable

§5.1 Higher order derivatives

§5.1.1 Leibnitz rule

Theorem 5.1 (Leibnitz rule)

If f and g are n-times differentiable functions, then the product fg is also n-times differentiable and its n^{th} derivative is given by

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)},$$

where $f^{(j)}$ is the j^{th} derivative of f with $f^{(0)} = f$.

Proof. We proceed by induction on n.

For $n = 1$, $(fg)' = f'g + fg'$. This proves the base case.

Assume for our induction hypothesis that the theorem holds for a fixed $n \in \mathbb{Z}^+$, i.e.

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}.$$

Then,

$$\begin{aligned}
(fg)^{(n+1)} &= \left[\sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}\right]' \\
&= \sum_{k=0}^{n} \binom{n}{k} f^{(n+1-k)} g^{(k)} + \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k+1)} \\
&= \sum_{k=0}^{n} \binom{n}{k} f^{(n+1-k)} g^{(k)} + \sum_{k=1}^{n+1} \binom{n}{k-1} f^{(n+1-k)} g^{(k)} \\
&= \binom{n}{0} f^{(n+1)} g^{(0)} + \sum_{k=1}^{n} \binom{n}{k} f^{(n+1-k)} g^{(k)} + \sum_{k=1}^{n} \binom{n}{k-1} f^{(n+1-k)} g^{(k)} + \binom{n}{n} f^{(0)} g^{(n+1)} \\
&= \binom{n+1}{0} f^{(n+1)} g^{(0)} + \left(\sum_{k=1}^{n} \left[\binom{n}{k-1} + \binom{n}{k}\right] f^{(n+1-k)} g^{(k)}\right) + \binom{n+1}{n+1} f^{(0)} g^{(n+1)} \\
&= \binom{n+1}{0} f^{(n+1)} g^{(0)} + \sum_{k=1}^{n} \binom{n+1}{k} f^{(n+1-k)} g^{(k)} + \binom{n+1}{n+1} f^{(0)} g^{(n+1)} \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} f^{(n+1-k)} g^{(k)}.
\end{aligned}$$

□

Proposition 5.1 (Standard results)

We state some basic results.

1. Let $y = x^k$, then
$y^{(n)} = \mathrm{D}^n y = k(k-1)\cdots(k-n+1)x^{k-n} = \frac{k!}{(k-n)!}x^{k-n} \quad \forall n \in \mathbb{Z}^+.$

   In particular,

   (i) if $n = k \in \mathbb{Z}^+$, then $y^{(k)} = k!$;

   (ii) if $k \in \mathbb{Z}^+ : k < n$, then $y^{(n)} = 0$;

   (iii) if $k \in \mathbb{Z}^+$ so that $-k \in \mathbb{Z}^-$ and $y = x^{-k}$, then

$$y^{(n)} = (-1)^n \frac{k!}{(k-n)!}\frac{1}{x^{k+n}}.$$

2. $y = \log(x) \quad (x > 0) \implies y^{(n)} = (-1)^n (n-1)! x^{-n}$.
3. $y = \frac{1}{x-a} \implies y^{(n)} = (-1)^n \frac{n!}{(x-a)^{n+1}}$.
4. $y = \frac{1}{ax+b} \implies y^{(n)} = (-1)^n \frac{n!}{(ax+b)^{n+1}} a^n$.

§5.2 Concavity and inflection points

Remark. We know that $f'(x) > 0 \implies f(x)$ is increasing and $f'(x) < 0 \implies f(x)$ is decreasing. Clearly then the $\operatorname{sgn}(f''(x))$ tells us whether f' is increasing or decreasing. If $\exists x = x_0 \in \mathbb{R} : f'(x_0) = 0$, then x_0 is a *critical point* of f.

If $f'(x_0) = 0$ then

1. $f''(x_0) < 0 \implies f(x)$ has a local maxima at $x = x_0$.
2. $f''(x_0) > 0 \implies f(x)$ has a local minima at $x = x_0$.

Even if $f'(x) \neq 0$ we can still extract some information about $f(x)$ using the second derivative.

Definition 5.1

concave

§5.3 Envelopes

§5.4 Curvature

§5.5 Asymptotes

§5.6 Integration by reduction formulas

Theorem 5.2 (Integration by reduction)

Let $m, n \in \mathbb{Z}$.

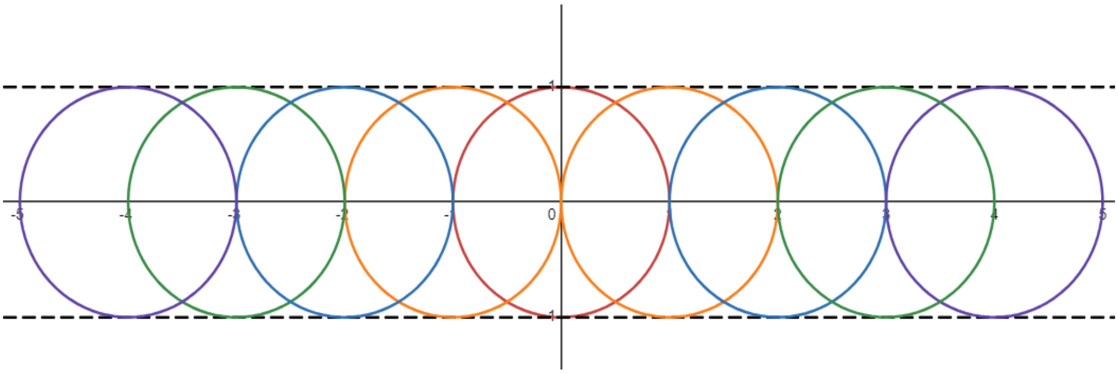

Figure 1: $(x-a)^2+y^2=1$.

1. If $I_n = \int \sin^n(x)\,\mathrm{d}x$ then
$$I_n = \frac{-\sin^{n-1}(x)\cos(x)}{n} + \frac{(n-1)I_{n-2}}{n}.$$

2. If $I_n = \int \cos^n(x)\,\mathrm{d}x$ then
$$I_n = \frac{\cos^{n-1}(x)\sin(x)}{n} + \frac{(n-1)I_{n-2}}{n}.$$

3. If $I_n = \int \tan^n(x)\,\mathrm{d}x$ then
$$I_n = \frac{\tan^{n-1}(x)}{n-1} - I_{n-2}.$$

4. If $I_n = \int \sec^n(x)\,\mathrm{d}x$ then
$$I_n = \frac{\sec^{n-2}(x)\tan(x)}{n-1} + \frac{(n-2)I_{n-2}}{n-1}.$$

5. If $I_{m,n} = \int \sin^m(x)\cos^n(x)\,\mathrm{d}x$ then
$$\begin{aligned} I_{m,n} &= \frac{\sin^{m+1}(x)\cos^{n-1}(x)}{m+n} + \frac{(n-1)I_{m,n-2}}{m+n} \\ &= \frac{-\sin^{m-1}(x)\cos^{n+1}(x)}{m+n} + \frac{(m-1)I_{m-2,n}}{m+n} \end{aligned}$$

Proof. 1. $I_n = \displaystyle\int \sin^n(x)\,\mathrm{d}x = \int \sin^{n-1}(x)\sin(x)\,\mathrm{d}x$
$$\begin{aligned} &= \sin^{n-1}(x)\int \sin(x)\,\mathrm{d}x - \int \frac{d}{\mathrm{d}x}\left(\sin^{n-1}(x)\right)\left(\int \sin(x)\,\mathrm{d}x\right)\,\mathrm{d}x \\ &= -\sin^{n-1}(x)\cos(x) - \int \cos(x)\left((n-1)\sin^{n-2}(x)\right)(-\cos(x))\,\mathrm{d}x \end{aligned}$$

$$= -\sin^{n-1}(x)\cos(x) + (n-1)\int \sin^{n-2}(x)\cos^2(x)\ \mathrm{d}x$$

$$= -\sin^{n-1}(x)\cos(x) + (n-1)\int \sin^{n-2}(x)\left(1-\sin^2(x)\right)\ \mathrm{d}x$$

$$= -\sin^{n-1}(x)\cos(x) + (n-1)\int \left(\sin^{n-2}(x) - \sin^n(x)\right)\ \mathrm{d}x$$

$$= -\sin^{n-1}(x)\cos(x) + (n-1)(I_{n-2} - I_n).$$

Thus, $nI_n = I_n + (n-1)I_n = -\sin^{n-1}(x)\cos(x) + (n-1)I_{n-2}$.

2. $I_n = \displaystyle\int \cos^n(x)\ \mathrm{d}x = \int \cos^{n-1}(x)\cos(x)\ \mathrm{d}x$

$$= \cos^{n-1}(x)\int \cos(x)\ \mathrm{d}x - \int \frac{d}{\mathrm{d}x}\left(\cos^{n-1}(x)\right)\left(\int \cos(x)\ \mathrm{d}x\right)\ \mathrm{d}x$$

$$= \cos^{n-1}(x)\sin(x) + \int \sin(x)\left((n-1)\cos^{n-2}(x)\right)\sin(x)\ \mathrm{d}x$$

$$= \cos^{n-1}(x)\sin(x) + (n-1)\int \cos^{n-2}(x)\sin^2(x)\ \mathrm{d}x$$

$$= \cos^{n-1}(x)\sin(x) + (n-1)\int \cos^{n-2}(x)\left(1-\cos^2(x)\right)\ \mathrm{d}x$$

$$= \cos^{n-1}(x)\sin(x) + (n-1)\int \left(\cos^{n-2}(x) - \cos^n(x)\right)\ \mathrm{d}x$$

$$= \cos^{n-1}(x)\sin(x) + (n-1)(I_{n-2} - I_n).$$

Thus, $nI_n = I_n + (n-1)I_n = \cos^{n-1}(x)\sin(x) + (n-1)I_{n-2}$.

3. $I_n = \displaystyle\int \tan^n(x)\ \mathrm{d}x = \int \tan^{n-2}(x)\tan^2(x)\ \mathrm{d}x$

$$= \int \overbrace{\tan^{n-2}(x)}^{t^{n-2}}\underbrace{\sec^2(x)\ \mathrm{d}x}_{\mathrm{d}t} - \int \tan^{n-2}(x)\ \mathrm{d}x$$

$$= \int t^{n-2}\ \mathrm{d}t - I_{n-2}$$

$$= \frac{t^{n-1}}{n-1} - I_{n-2}$$

$$= \frac{\tan^{n-1}(x)}{n-1} - I_{n-2}.$$

4. $I_n = \displaystyle\int \sec^n(x)\ \mathrm{d}x = \int \sec^{n-2}(x)\sec^2(x)\ \mathrm{d}x$

$$= \sec^{n-1}(x)\int \sec^2(x)\ \mathrm{d}x - \int \frac{d}{\mathrm{d}x}(\sec^{n-1}(x))\left(\int \sec^2(x)\ \mathrm{d}x\right)\ \mathrm{d}x$$

$$= \sec^{n-1}(x)\tan(x) - (n-2)\int \sec^{n-3}(x)\sec(x)\tan(x)\tan(x)\ \mathrm{d}x$$

$$= \sec^{n-1}(x)\tan(x) - (n-2)\int \sec^{n-2}(x)\tan^2(x)\,\mathrm{d}x$$

$$= \sec^{n-1}(x)\tan(x) - (n-2)\int \sec^{n-2}(x)(\sec^2(x)-1)\,\mathrm{d}x$$

$$= \sec^{n-1}(x)\tan(x) - (n-2)\int \left(\sec^{n}(x)-\sec^{n-2}(x)\right)\,\mathrm{d}x$$

$$= \sec^{n-1}(x)\tan(x) - (n-2)\int \left(I_n - I_{n-2}\right).$$

Thus,

$$(n-1)I_n = I_n + (n-2)I_n = \sec^{n-1}(x)\tan(x) + (n-2)I_{n-2}.$$

5. $\int \sin^m(x)\cos^n(x)\,\mathrm{d}x$

$$= \int \sin^{m-1}(x)\cos^n(x)\sin(x)\,\mathrm{d}x$$

$$= \sin^{m-1}(x)\int \cos^n(x)\sin(x)\,\mathrm{d}x - \int \frac{d}{\mathrm{d}x}\left(\sin^{m-1}(x)\right)\left(\int \cos^n(x)\sin(x)\,\mathrm{d}x\right)\,\mathrm{d}x$$

□

§5.7 Parametric equations

§5.8 Parameterizing a curve

§5.9 Arc length of a curve

§5.10 Arc length of parametric curves

§5.11 Area under a curve

§5.12 Area and volume of surface of revolution.