Algebraic Number Theory

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Final exam

Question 1. Define a Dedekind domain. State the Chinese Remainder Theorem for Dedekind domains. Prove that a Dedekind domain with finitely many prime ideals is a principal ideal domain.

Question 2. State the Minkowski bound for a number field. Hence compute the ring of integers and the ideal class group of $\mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{-5})$ and $\mathbb{Q}(\sqrt{5})$.

Question 3. Let α be a root of $X^3 + X + 1 = 0$. Then show that $\{1, \alpha, \alpha^2\}$ is an integral basis of $\mathbb{Q}(\alpha)$.

Question 4. Compute the discriminant of $\mathbb{Q}(\sqrt{2})$ and hence for every rational prime p find the prime decomposition of p modulo the discriminant of $\mathbb{Q}(\sqrt{2})$.

Question 5. Find a fundamental unit in (i) $\mathbb{Q}(\sqrt{6})$ and (ii) $\mathbb{Q}(\sqrt{5})$.

Question 6. Are $\dfrac{\sqrt{5}+\sqrt{17}}{2}$ and $\dfrac{\sqrt[3]{2}+1}{3}$ algebraic integers ? Justify.

Question 7. Show that $X^3 - 6 = 0$ has a root in $\mathbb{Q}_7$.

Question 8. Let $\| \cdot \|$ be a non-Archimedean valuation on the field $\mathbb{Q}$. Show that $\|m\| \leq 1$ for every $m \in \mathbb{Z}$.

Question 9. Find all primes that are totally ramified in the number field $\mathbb{Q}(\sqrt{5}, \sqrt{17})$.