Ex/SC/MATH/UG/MAJOR/TH/21/206/2025

Bachelor of Science Examination, 2025

(2^{nd} Year, 1^{st} Semester)

MATHEMATICS

Paper : Major – 06 (Ring Theory)

Time : *Two Hours* *Full Marks* : 40

Notations / Symbols have their usual meaning.

CO 1 : Answer *any* **one** question. 10×1=10

1. *(a)* Let R and R' be two rings with identities 1_R and $1_{R'}$ respectively and $f : R \to R'$ be a ring epimorphism. Show that $f(1_R) = 1_{R'}$. Does this result hold if f is a ring homomorphism? Justify. 3+2

   (b) Define kernel of a ring homomorphism. Let I be a nonempty subset of a ring R. Show that I is an ideal of R if and only if I is the kernel of some ring homomorphism. 1+4

2. *(a)* Let R be a commutative ring with identity 1_R and R' be an integral domain with identity $1_{R'}$. If $f : R \to R'$ be a non-zero homomorphism then show that $f(1_R) = 1_{R'}$. Does this result hold if R' is not an integral domain? Justify. 3+2

(*b*) Define field. Show that any integral domain with only a finite number of ideals is a field. Hence conclude that any finite integral domain is a field. 1+3+1

CO 2 : Answer *any* **one** question. 10×1=10

3. (*a*) Define quotient ring. Show that quotient ring of a commutative ring is commutative. Let R be a ring and I be an ideal of R such that the quotient ring R/I is a commutative ring. Is R a commutative ring? Justify. 2+1+2

(*b*) State and prove first isomorphism theorem of ring. 2+3

4. (*a*) Let R be a commutative ring with identity and P be an ideal of R. When is P said to be a prime ideal of R? Show that P is a prime ideal of R if and only if R/P is an integral domain. 1+4

(*b*) Define maximal ideal of a ring. Let R be a Boolean ring with identity and M be a proper ideal of R. Show that M is a maximal ideal of R if and only if M is a prime ideal of R. 1+4

CO 3 : Answer *any* **one** question. 10×1=10

5. (*a*) Define PID. Show that every ED is a PID. 1+4

(*b*) Let R be a ring with identity such that $\mathbb{Z} \subset R \subset \mathbb{Q}$. Show that R is a PID. Is R a field? Justify. 4+1

6. (*a*) Define prime and irreducible element in a ring. Show that in an integral domain every prime element is irreducible. 2+3

(*b*) Find the units of $\mathbb{Z}[i]$. Show that the set of the units of $\mathbb{Z}[i]$ forms a cyclic group w.r.t. multiplication. Is 5 an irreducible element of $\mathbb{Z}[i]$? Justify. 2+2+1

CO 4 : Answer *any* **one** question. 10×1=10

7. (*a*) Let R be a commutative ring with identity. Show that $R[x]/<x>\cong R$. Hence conclude that $<x>$ is a prime ideal of $\mathbb{Z}[x]$ but not a maximal ideal of $\mathbb{Z}[x]$. 3+2

(*b*) Let F be an infinite field and $f(x), g(x) \in F[x]$ such that $f(a) = g(a)$ for all $a \in F$. Show that $f(x) = g(x)$. Does this result hold if F is a finite field? Justify. Let $a \in F$ be a root of $f(x) \in F[x]$. Does $f(x) \in <x-a>$? Justify. 2+2+1

8. (*a*) Construct the field of fractions $Q(R)$ of an integral domain R. Let R_1, R_2 be two integral domains and $Q(R_1), Q(R_2)$ be the field of fractions of R_1, R_2 respectively such that $Q(R_1) \cong Q(R_2)$. Is $R_1 \cong R_2$? Justify. 3+2

(*b*) State Eisenstein's irreducibility criterion for a polynomial. Let $f(x) = x^3 - 312312x + 123123 \in \mathbb{Z}[x]$. Is $f(x)$ irreducible in $\mathbb{Z}[x]$? Justify. 2+3

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