Module Theory

Semester 7, 2026

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$$
\begin{tikzcd}
 & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' &
\end{tikzcd}
$$

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List of Lectures

1. Modules & Categories

1.1 Modules

A vector space is a field action whereas a left (right) module is a left (right) ring action. A field is commutative so left and right vector spaces are the same.

LECTURE 1
21^{st} Jul, 2026

Definition 1: Vector space

Let F be a field and V be a nonempty set. Then V is a **vector space** over F if

(i) there exists an addition $+$ on V s.t. $(V, +)$ is an abelian group; &

(ii) there exists a mapping[a]

$$\begin{aligned} F \times V &\longrightarrow V \\ (\lambda, v) &\longmapsto \lambda v \end{aligned}$$

satisfying

(a) $\lambda(v + w) = \lambda v + \lambda w$ for all $\lambda \in F$ and $v, w \in V$;
(b) $(\lambda + \mu)v = \lambda v + \mu v$ for all $\lambda, \mu \in F$ and $v \in V$;
(c) $(\lambda\mu)v = \lambda(\mu v)$ for all $\lambda, \mu \in F$ and $v \in V$; and
(d) $1_F v = v$ for all $v \in V$, where 1_F is the multiplicative identity of F.

[a] field action

Definition 2: Left module

Let R be a ring and M be a nonempty set. Then M is a **left module** over R, or a **left-R module** if

(i) there exists an addition $+$ on M s.t. $(M, +)$ is an abelian group; &

(ii) there exists a mapping[a]

$$\begin{aligned} R \times M &\longrightarrow M \\ (r, m) &\longmapsto rm \end{aligned}$$

satisfying

(a) $r(m + n) = rm + rn$ for all $r \in R$ and $m, n \in M$;
(b) $(r + s)m = rm + sm$ for all $r, s \in R$ and $m \in M$;
(c) $(rs)m = r(sm)$ for all $r, s \in R$ and $m \in M$; and

[a] left ring action

Similarly, one can define a right R-module by considering the right action $M \times R \to M$ instead.

Note. For a left R-module M, in axiom (c), $(rs)m = r(sm)$ need not be equal to $s(rm)$, $r, s \in R$, $m \in M$. For a right R-module M,

$$m(rs) = (mr)s \neq (ms)r.$$

Differences between vector space and module

(i) Unless the ring is commutative (for a module), the left action (for a module) and the right action are different. But for a vector space over a field, the left action gives rise to the right action and vice versa.

(ii) A ring may not have an identity and in that case there is no meaning of the analogous defining condition (d) of Definition 1 for a module. If the ring R has an identity 1_R, and $1_R m = m$ for all $m \in M$, then M is called a **unitary** or **unital left R-module**.

If the ring is commutative, then the notion of a left and right R-module is the same. In this case, we simply call it an *R-module*. In fact, whenever it is clear, we will say R-module to mean *left* R-module if R is not necessarily commutative.

Suppose G is an (additive) abelian group. Recall that for any $n \in \mathbb{Z}$ and $a \in G$,

$$na := \begin{cases} a + a + \cdots + a \quad (n \text{ times}), & \text{if } n > 0 \\ (-a) + (-a) + \cdots + (-a) \quad (|n| \text{ times}), & \text{if } n < 0 \\ 0_G, & \text{if } n = 0. \end{cases}$$

The above formula for na is in fact an action of $\mathbb{Z}$ on G, i.e.,

$$\begin{aligned} \mathbb{Z} \times G &\longrightarrow G \\ (n, a) &\longmapsto na \end{aligned}$$

such that the following properties are satisfied

(i) $n(a + b) = na + nb$ for all $n \in \mathbb{Z}$ and $a, b \in G$;

(ii) $(m + n)a = ma + na$ for all $m, n \in \mathbb{Z}$ and $a \in G$; and

(iii) $(mn)a = m(na)$ for all $m, n \in \mathbb{Z}$ and $a \in G$.

(iv) $1a = a$ for all $a \in G$.

Thus, G is a (unitary) $\mathbb{Z}$-module.

Note. Thus any (additive) abelian group is canonically a $\mathbb{Z}$-module. Conversely, any $\mathbb{Z}$-module is an (additive) abelian group.

Remark. One can see a module as a generalisation of an (additive) abelian group (just by replacing the natural $\mathbb{Z}$-action by an R-action for an arbitrary ring R) or as a generalisation of a vector space over a field (by replacing the action of a field F by the action of an arbitrary ring R).

Ring of endomorphisms of an abelian group

Question 1. *Suppose $(M, +)$ is an abelian group and R is a ring. Is there any guarantee that R has an action (left or right) on M s.t. M becomes an R-module ?*

To answer the above question it may help to consider the following question:

Question 2. *What is the difference or connection between a group action and a ring action (giving rise to a module) ? OR Is there any analogy between the two ?*

Recall: If a group has an action on a set S then there exists a group homomorphism

$$\begin{aligned} G &\longrightarrow \mathrm{Sym}(S) \\ a &\longmapsto \phi_a : S \to S \\ &\qquad\quad x \to ax \end{aligned}$$

induced by the action of G on S (often denoted $G \curvearrowright S$).

Conversely, if there exists a group homomorphism f from G to the canonical group $\mathrm{Sym}(S)$

$$\begin{aligned} f : G &\longrightarrow \mathrm{Sym}(S) \\ g &\longmapsto f(g) : S \to S \end{aligned}$$

then

$$\begin{aligned} G \times S &\longrightarrow \mathrm{Sym}(S) \\ (g, s) &\longmapsto gs := f(g)(s) \end{aligned}$$

defines an action of G on S. G always has an action on S, the trivial action

$$\begin{aligned} G \times S &\to S \\ (g, s) &\mapsto gs = s \end{aligned}$$

Similar to how $\mathrm{Sym}(S)$ is the *canonical group*, we want to find an analogous *canonical ring*.

Recall: Suppose $(M,+)$ is an abelian group. Then

$$\mathrm{End}(M) = \mathrm{Hom}(M,M) = \text{ the set of all group endomorphisms of } M$$

is a ring, where $+$ and $\cdot$ are defined as follows: for all $f,g \in \mathrm{End}(M)$,

$$\begin{aligned} f+g :& M \to M \\ x &\mapsto (f+g)(x) \\ (f+g)(x) &:= f(x)+g(x) \quad \forall x \in M \end{aligned}$$

and

$$\begin{aligned} fg :& M \to M \\ x &\mapsto (fg)(x) \\ (fg)(x) &:= f(g(x)) \quad \forall x \in M \end{aligned}$$

i.e. $fg = f \circ g$.

Exercise 1. *Prove that $f+g, fg \in \mathrm{End}(M)$. Hence, prove that $(\mathrm{End}(M),+,\cdot)$ is a ring.*

This ring $\mathrm{End}(M)$ is the *canonical ring*.

Exercise 2. *Suppose M is a (left) R-module. Prove that $f : R \to \mathrm{End}(M)$ defined by*

$$\begin{aligned} f(r) :& M \to M \\ m &\mapsto f(r)(m) := rm \end{aligned}$$

for all $r \in R$, $m \in M$; is a ring homomorphism.

Exercise 3. *Suppose M is an (additive) abelian group & R is a ring s.t. $\phi : R \to \mathrm{End}(M)$ is a ring homomorphism. Prove that with this action of R on M, M becomes a left R-module.*

The trivial action

$$\begin{aligned} R \times M &\to M \\ (r,m) &\mapsto rm = 0 \end{aligned}$$

satisfies all the defining conditions of a left R-module but not the defining condition of a unitary module.

So, just as group action is representation of a group by permutation, module is representation of a ring as an endomorphism. To summarise: a group action $G \curvearrowright S$ induces a group homomorphism

LECTURE 2
24^{th} Jul, 2026

$$G \curvearrowright S \longmapsto \phi \in \mathrm{Hom}(G, \mathrm{Sym}(S))$$

where $\phi(g)(s) = \phi_g(s) = gs$; but what about the converse, i.e.

$$G \curvearrowright S \overset{?}{\longleftarrow} \psi \in \mathrm{Hom}(G, \mathrm{Sym}(S))$$

The answer is yes (take $\psi = \phi$) and so $G \curvearrowright S \longleftrightarrow \phi \in \mathrm{Hom}(G, \mathrm{Sym}(S))$. Similar to how $\mathrm{Sym}(S)$ is the canonical group, $\mathrm{End}(M,+,\cdot)$ is the canonical ring where $(M,+)$ is an abelian group and $\mathrm{End}(M) = \mathrm{Hom}(M,M)$ is the set of all group endomorphisms on M. Then

$$\begin{aligned} \psi :& R \longrightarrow \mathrm{End}(M) \\ r &\longmapsto \psi(r) : M \longrightarrow M \\ & \qquad\qquad\quad m \longmapsto rm \end{aligned}$$

is a ring homomorphism.

Group representations

(1) Permutation representation $G \curvearrowright \mathrm{Sym}(S)$;

(2) Linear representation $G \curvearrowright \mathrm{GL}(V)$, where $\mathrm{GL}(V)$ is the set of all bijective linear operators on V (endomorphisms on V); and

(3) Matrix representation $G \curvearrowright \mathrm{GL}_n(F)$, where F is a field.

The permutation representations give rise to linear representations, and thus also matrix representations, of a group. Suppose $G \curvearrowright \mathrm{Sym}(S)$. Then there exists a vector space V over a field F s.t. any basis $\mathcal{B}$ of V has the property that $|\mathcal{B}| = |S|$.
Suppose $\mathcal{B} = \{v_i\}_{i\in S}$. Let $\sigma \in \mathrm{Sym}(S)$. Then there exists (why ?) a linear operator

$$\begin{aligned} T_\sigma : V &\longrightarrow V \\ v_i &\longmapsto v_{\sigma(i)} \end{aligned}$$

then $\{T_\sigma(v_i)\}_{i\in S} = \{v_{\sigma(i)}\}_{i\in S}$ is a basis of V. Since T_σ maps the basis $\{v_i\}_{i\in S}$ to the basis $\{v_{\sigma(i)}\}_{i\in S}$, therefore $T_\sigma \in \mathrm{GL}(V)$. Hence, we get a correspondence

$$\begin{array}{ccccc} G & \xrightarrow{\psi} & \mathrm{Sym}(S) & \longrightarrow & \mathrm{GL}(V) \\ g & \longmapsto & \psi_g = \sigma & \longmapsto & T_\sigma \end{array}$$

Here $\dim V = |G|$. Recall how we proved Cayley's theorem:

Theorem 1: Cayley

Every group G is isomorphic to a subgroup of the symmetric group $\mathrm{Sym}(G)$.

Proof. We consider the action of G on itself,

$$\begin{aligned} G &\longrightarrow \mathrm{Sym}(G) \\ g &\longmapsto \lambda_g : G \longrightarrow G \\ &\qquad\qquad a \longmapsto ga \end{aligned} \tag{1.1}$$

which induces a group homomorphism $\phi(g) = \lambda_g$ on G. Let $g \in \ker\phi$, then $g = ge = \lambda_g(e) = e$ so $\ker\phi = \{e\}$ and the action is faithful. Thus, by the first isomorphism theorem, $\mathrm{im}\,\phi \cong G/\ker\phi = G/\{e\} = G$. ∎

The permutation representation $\phi(g) = \lambda_g$ above in Eq. (1.1) is called a *regular representation*.
Take the Klein 4-group $V_4 = \{e, a, b, c\}$ defined by the Cayley table

$\cdot$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

and consider the left multiplication actions by each element,

$$\begin{aligned} \lambda_e : e &\longmapsto e \\ a &\longmapsto ea = a \\ b &\longmapsto eb = b \\ c &\longmapsto ec = c \\ \lambda_a : e &\longmapsto a \\ a &\longmapsto e \\ b &\longmapsto ab = c \\ c &\longmapsto ac = b \\ \lambda_b : e &\longmapsto b \\ a &\longmapsto ba = c \\ b &\longmapsto e \\ c &\longmapsto bc = a \\ \lambda_c : e &\longmapsto c \\ a &\longmapsto ca = b \\ b &\longmapsto cb = a \\ c &\longmapsto e \end{aligned}$$

$$\lambda_e = \begin{pmatrix} e & a & b & c \\ e & a & b & c \end{pmatrix}$$

$$\lambda_a = \begin{pmatrix} e & a & b & c \\ a & e & c & b \end{pmatrix}$$

$$\lambda_b = \begin{pmatrix} e & a & b & c \\ b & c & e & a \end{pmatrix}$$

$$\lambda_c = \begin{pmatrix} e & a & b & c \\ c & b & a & e \end{pmatrix}$$

$$\{\lambda_e, \lambda_a, \lambda_b, \lambda_c\} \subseteq \mathrm{Sym}(G) \cong S_4$$

$$\begin{aligned} \lambda_e &\mapsto (1) = (2) = (3) = (4) \\ \lambda_a &\mapsto (1 \quad 2)(3 \quad 4) \\ \lambda_b &\mapsto (1 \quad 3)(2 \quad 4) \\ \lambda_c &\mapsto (1 \quad 4)(2 \quad 3) \end{aligned}$$

Let V be a vector space of dimension $|V_4| = 4$. Let $\{v_1, v_2, v_3, v_4\}$ be an ordered basis. Then

$$\begin{aligned} T_{\lambda_e} :& V \to V \\ & v_i \mapsto v_{\lambda_e(i)} = v_i \end{aligned}$$

and $[T_{\lambda_e}]_{\mathcal{B}} = I_4$ where $\mathcal{B} = \{e_1, e_2, e_3, e_4\}$ is the standard ordered basis. Similarly,

$$\begin{aligned} T_{\lambda_a} :& V \to V \\ & v_i \mapsto v_{\lambda_a(i)} \\ & \begin{pmatrix} v_1 & v_2 & v_3 & v_4 \\ v_2 & v_1 & v_4 & v_3 \end{pmatrix} \end{aligned} \qquad \begin{aligned} T_{\lambda_b} :& V \to V \\ & v_i \mapsto v_{\lambda_b(i)} \\ & \begin{pmatrix} v_1 & v_2 & v_3 & v_4 \\ v_3 & v_4 & v_1 & v_2 \end{pmatrix} \end{aligned} \qquad \begin{aligned} T_{\lambda_c} :& V \to V \\ & v_i \mapsto v_{\lambda_c(i)} \\ & \begin{pmatrix} v_1 & v_2 & v_3 & v_4 \\ v_4 & v_3 & v_2 & v_1 \end{pmatrix} \end{aligned}$$

Hence, $A = [T_{\lambda_a}]_{\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, $B = [T_{\lambda_b}]_{\mathcal{B}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ and $C = [T_{\lambda_c}]_{\mathcal{B}} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$. Hence, $V_4 \cong \{I_4, A, B, C\} \leqslant \mathrm{GL}_4(F)$.

Ring representations

Definition 3: Representation of a ring

Suppose R is a ring. Then any ring homomorphism

$$\psi : R \to \mathrm{End}(M)$$

is called a **representation** of the ring R; here M is an (additive) abelian group, and $\mathrm{End}(M)$ is the ring of group endomorphisms of M.

Exercise 4. *Suppose $\psi : R \to \mathrm{End}(M)$ is a ring homomorphism. Prove that*

$$\begin{aligned} R \times M &\to M \\ (r, m) &\mapsto rm := \psi(r)(m) \end{aligned}$$

defines a left R-module M s.t. the induced ring homomorphism is ψ.

Proof. Task: Routine verification. ∎

Question 3. *Suppose R is a ring & M is an (additive) abelian group. Does there exist an action of R on M which makes M into an R-module ? OR (same question) Does there exist a ring homomorphism $R \to \mathrm{End}(M)$?*

Every abelian group is a (unitary) $\mathbb{Z}$-module in exactly one way

Consider the following problem from [Jac85, §3.2, p. 6, page 166]: Suppose $(M, +)$ is an abelian group. Prove that M can be made into a unitary $\mathbb{Z}$-module in exactly one way. In other words, there is only one ring homomorphism $\mathbb{Z} \to \mathrm{End}(M)$ which maps identity to identity.

In fact, there is only one ring homomorphism $\mathbb{Z} \to R$ (R a ring with identity 1) that maps 1 to 1. Let $f : \mathbb{Z} \to R$ (R a ring with identity 1) be a ring homomorphism s.t. $f(1) = 1$. Then, for any $n \in \mathbb{Z}$, $f(n) = nf(1)$. So,

$$\begin{aligned} \psi :& \mathbb{Z} \to \mathrm{End}(M) \\ & 1 \mapsto \psi(1) : M \to M \\ & \qquad\qquad a \mapsto a \\ & n \mapsto \psi(n) : M \to M \\ & \qquad\qquad a \mapsto na \end{aligned}$$

Hence, any (additive) abelian group M can be made into a unitary $\mathbb{Z}$-module in exactly one way.

1.2 Categories

Lecture 3
28th Jul, 2026

Definition 4

A *category* $\mathscr{C}$ consists of a *class* $\mathrm{ob}(\mathscr{C})$ of *objects*, usually denoted by $A, B, C, \ldots$, & *sets* of *morphisms* (or *arrows*) between objects, usually denoted by $\mathrm{Hom}(A, B)$ or $\mathrm{Mor}(A, B)$ (where A, B are objects of $\mathscr{C}$) s.t. the following hold:

(i) for $A, B, C \in \mathrm{ob}(\mathscr{C})$ there exists a function

$$\begin{aligned} \mathrm{Hom}(A,B) \times \mathrm{Hom}(B,C) &\longrightarrow \mathrm{Hom}(A,C) \\ (f,g) &\longmapsto g \circ f \end{aligned}$$

satisfying $(f \circ g) \circ h = f \circ (g \circ h)$, *wherever all are defined*;

(ii) for each $A \in \mathrm{ob}\,\mathscr{C}$ there exists a distinguished element $\mathbb{1}_A \in \mathrm{Hom}(A, A)$ satisfying

$$f \circ \mathbb{1}_A = f \quad \& \quad \mathbb{1}_A \circ g = g \quad \forall f \in \mathrm{Hom}(A,B),\ g \in \mathrm{Hom}(C,A).$$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend left, "g\circ f"] & B \arrow[r, "g"] & C \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[rrr, bend left=40, "(f\circ g)\circ h"] \arrow[rr, bend right, "g\circ h"'] \arrow[rrr, bend right=40, "f\circ(g\circ h)"'] & B \arrow[r, "g"] \arrow[rr, bend left, "f\circ g"] & C \arrow[r, "f"] & D \end{tikzcd}$$

$$\begin{tikzcd} C \arrow[r, "g"] & A \arrow[r, "\mathbb{1}_A"] & A \arrow[r, "f"] & B \end{tikzcd}$$

Exercise 5. *Prove that $\mathbb{1}_A$, with the above property (ii), is unique for any $A \in \mathrm{ob}(\mathscr{C})$.*

We take a *class* of objects to avoid set-theoretic problems such as Russell's paradox. When the class $\mathrm{ob}(\mathscr{C})$ is a *set*, $\mathscr{C}$ is called a *small category*. Our Definition 4 for a category is not entirely precise but it suffices for now. The following are *concrete categories*,

Example 1

Grp: objects are groups, morphisms are group homomorphisms between two groups.

Example 2

For a ring R, $R-$**Mod** (or **Mod**$-R$): objects are left (or right) R-modules and morphisms are module homomorphisms.

Example 3

Set: objects are sets, morphisms are mappings between sets.

Example 4

Top: objects are topological spaces, morphisms are continuous mappings between topological spaces.

The following is an *abstract category*,

Example 5

Mon: $\mathrm{ob}(\mathbf{Mon})$ consists of a single object A (say) and $\mathbb{1}_A \in \mathrm{Hom}(A, A)$. For all $f, g, h \in \mathrm{Hom}(A, A)$, we have

$$\begin{tikzcd} A \arrow[loop above] \arrow[loop left] \arrow[loop below] \end{tikzcd} \qquad f \circ (g \circ h) = (f \circ g) \circ h$$

and $f \circ \mathbb{1}_A = \mathbb{1}_A \circ f = f$. We identify $\mathbb{1}_A \equiv A$. So $\mathrm{Hom}(A, A)$ is a monoid and this category **Mon** is called the *monoid category*. Conversely take a monoid $(M, \cdot)$ and identify $\mathbb{1}_M$ with M then this becomes a category. Thus, a single object category is nothing but a monoid.

Inverse of an arrow/morphism in a category

Definition 5: Inverse of an arrow/morphism

$$\begin{tikzcd} A \arrow[r, "f"'] & B \arrow[l, bend right, "g"'] \end{tikzcd} \qquad \text{If } f \circ g = \mathbb{1}_B \quad \& \quad g \circ f = \mathbb{1}_A, \text{ then } g = f^{-1}.$$

Example 6

A one object category in which every arrow has an inverse is nothing but a group.

Example 7

A category $\mathscr{C}$, s.t. every arrow/morphism has an inverse, is called a *groupoid*.[a] This is an example of an abstract category. A bundle of groups is a groupoid, but not the converse.

[a] To avoid confusion, we call a nonempty set equipped with a binary operation a *magma* instead of groupoid.

Counterexample 1

Let G be a directed graph, i.e. $G = (G^0, G^1, r, s)$ where G^0 is the set of vertices, G^1 is the set of edges, r is the range map and s is the source map; for $f \in G^1$

$$\begin{aligned} r :& G^1 \to G^0 \\ & f \mapsto r(f) \\ s :& G^1 \to G^0 \\ & f \mapsto s(f) \end{aligned} \qquad s(f) \bullet \xrightarrow{\quad f \quad} \bullet\, r(f)$$

although this has similarities with a category as defined in Definition 4, it fails to be one. This is because the composition of two edges need not exist,

$$v_1 \bullet \xrightarrow{\quad f \quad} v_2 \bullet \xrightarrow{\quad g \quad} v_3 \bullet$$

just because there is an edge between v_1 and v_2 and an edge between v_2 and v_3 does not mean that there is an edge between v_1 and v_3.

If we consider paths instead of edges as arrows, then we can make a directed graph into a category.

Example 8

Path category of a directed graph is an abstract category, where the objects are the vertices and morphisms are the paths. Identity is taken to be path of length 0, and every vertex is a path of length 0, every edge is a path of length 1. The composition of two paths is defined to be the concatenation of the two paths.

Correspondence between categories (functor)

We want to find a correspondence between the categories $R-$**Mod** and **Ab**, where $R-$**Mod** is the category of left R-modules (morphisms are module homomorphisms) and **Ab** is the category of abelian groups (morphisms are group homomorphisms).

Let M, N be R-modules and $\text{Hom}(M, N) = \text{Hom}_R(M, N) =$ set of all R-module homomorphisms from M to N.

Recall: For vector spaces V, W over a field F, $\mathcal{L}(V, W) :=$ set of all F-linear maps from V to W is a vector space over F. For $T_1, T_2 \in \mathcal{L}(V, W)$ & $\lambda \in F$,

$$\begin{aligned} T_1 + T_2 :& V \longrightarrow W \\ & v \longmapsto (T_1 + T_2)(v) := T_1(v) + T_2(v) \end{aligned} \quad \& \quad \begin{aligned} \lambda T_1 :& V \longrightarrow W \\ & v \longmapsto (\lambda T_1)(v) := \lambda T_1(v) \end{aligned}$$

Define for $f, g \in \mathrm{Hom}(M,N)$ & $\lambda \in R$,

$$\begin{aligned} f+g :& M \longrightarrow N \\ & m \longmapsto (f+g)(m) := f(m)+g(m) \end{aligned} \quad \& \quad \begin{aligned} \lambda f :& M \longrightarrow N \\ & m \longmapsto (\lambda f)(m) := \lambda f(m) \end{aligned}$$

Question 4. *Is* $\mathrm{Hom}(M,N)$ *a left R-module with the addition and multiplication defined above ?*

$f+g \in \mathrm{Hom}(M,N)$ for all $f,g \in \mathrm{Hom}(M,N)$ and $(\mathrm{Hom}(M,N),+)$ is an abelian group. But for all $\lambda \in R$ & $f \in \mathrm{Hom}(M,N)$, λf need not belong to $\mathrm{Hom}(M,N)$.

$$\begin{aligned} (\lambda f)(m_1+m_2) &= \lambda(f(m_1)+f(m_2)) \\ &= \lambda f(m_1) + \lambda f(m_2) \\ &= (\lambda f)(m_1) + (\lambda f)(m_2) \\ \text{but } (\lambda f)(\mu m_1) &= \lambda(f(\mu m_1)) \\ &= \lambda(\mu f(m_1)) \\ &\neq \mu(\lambda f(m_1)) \text{ as } R \text{ may not be commutative.} \end{aligned}$$

So all we can say is that $(\mathrm{Hom}(M,N),+)$ is an abelian group.

Let M be an R-module.

$$\begin{aligned} R\!-\!\mathbf{Mod} &\xrightarrow{\mathrm{Hom}(M,-)} \mathbf{Ab} \\ N &\longmapsto \mathrm{Hom}(M,N) \end{aligned}$$

$$\begin{tikzcd} & \mathrm{Hom}(M,N_2) \\ N_1 \arrow[r, "f"] & N_2 \\ & \mathrm{Hom}(M,N_1) \arrow[uu, "\mathrm{Hom}(M,f)"'] \end{tikzcd}$$

$$\begin{aligned} \mathrm{Hom}(M,N_1) &\xrightarrow{\mathrm{Hom}(M,f)} \mathrm{Hom}(M,N_2) \\ g &\longmapsto f \circ g \end{aligned}$$

$$(\mathrm{Hom}(M,f))(g) := f \circ g$$

$$\begin{tikzcd} & M \arrow[dl, "g"'] \arrow[dr, "f\circ g"] & \\ N_1 \arrow[rr, "f"'] & & N_2 \end{tikzcd}$$

$$\begin{tikzcd} N_1 \arrow[r, "f"] \arrow[rr, bend left, "g\circ f"] & N_2 \arrow[r, "g"] & N_3 \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Hom}(M,N_1) \arrow[r, "\mathrm{Hom}(M,f)"] \arrow[dr, "\mathrm{Hom}(M,\, g\circ f)"'] & \mathrm{Hom}(M,N_2) \arrow[d, "\mathrm{Hom}(M,g)"] \\ & \mathrm{Hom}(M,N_3) \end{tikzcd}$$

We decided what direction the arrow should be between $\mathrm{Hom}(M,N_1)$ and $\mathrm{Hom}(M,N_2)$ by looking at the directions of the arrows between N_1, N_2 and M. We also have an identity arrow $\mathbb{1}_{\mathrm{Hom}(M,N_1)}$ on each object $\mathrm{Hom}(M,N_1)$ in the category $\mathbf{Ab}$ given by

$$N_1 \xrightarrow{\mathbb{1}_{N_1}} N_2$$

$$\mathrm{Hom}(M,N_1) \xrightarrow{\mathrm{Hom}(M,\mathbb{1}_{N_1})} \mathrm{Hom}(M,N_2)$$

$$\mathbb{1}_{\mathrm{Hom}(M,N_1)} = \mathrm{Hom}(M,\mathbb{1}_{N_1})$$

The above type of correspondence between categories is called a *functor*.

Lecture 4
31st Jul, 2026

Precise definition of a category

We now give a precise definition of a category. Compare the following definition with Definition 4

Definition 6: Category

A **category** $\mathscr{C}$ consists of a *class* $\mathrm{ob}(\mathscr{C})$ of *objects*, usually denoted by $A, B, C, \ldots$ & a *class* $\mathrm{Mor}(\mathscr{C})$ of *morphisms* (or *arrows*) between objects, whose elements are usually denoted by $f, g, \ldots$ together with two mappings

$$\mathrm{dom} : \mathrm{Mor}(\mathscr{C}) \to \mathrm{ob}\,\mathscr{C}$$
$$\mathrm{cod} : \mathrm{Mor}(\mathscr{C}) \to \mathrm{ob}\,\mathscr{C}$$

$$\mathrm{dom} f = A \xrightarrow{\quad f \quad} \mathrm{cod} f = B \quad (\text{may be written as } f : A \to B)$$

& a mapping

$$\mathrm{ob}\,\mathscr{C} \longrightarrow \mathrm{Mor}(\mathscr{C})$$
$$A \longmapsto \mathbb{1}_A : A \to A$$

& s.t. there exists a *partial mapping/binary operation*

$$\mathrm{Mor}(\mathscr{C}) \times \mathrm{Mor}(\mathscr{C}) \longrightarrow \mathrm{Mor}(\mathscr{C})$$
$$(f, g) \longmapsto f \circ g \quad \text{iff } \mathrm{cod}\, g = \mathrm{dom}\, f$$

satisfying

$$(f \circ g) \circ h = f \circ (g \circ h) \quad \forall f, g, h \in \mathrm{Mor}(\mathscr{C}) \tag{1.2}$$

&

$$\mathbb{1}_A \circ f = f \text{ and } f \circ \mathbb{1}_A = f \quad \forall f \in \mathrm{Mor}(\mathscr{C}) \tag{1.3}$$

whenever the compositions are defined.

If $\mathrm{Mor}(\mathscr{C})$ is a set, then $\mathscr{C}$ is called a **locally small category**. If both $\mathrm{ob}(\mathscr{C})$ & $\mathrm{Mor}(\mathscr{C})$ are sets, then $\mathscr{C}$ is called a **small category**.

$$\begin{tikzcd} \mathrm{dom} g \bullet \arrow[r, "g"] \arrow[rr, bend left, "f \circ g"] & \overset{\mathrm{cod} g}{\underset{\mathrm{dom} f}{\bullet}} \arrow[r, "f"] & \bullet\ \mathrm{cod} f \end{tikzcd}$$

$$\begin{tikzcd} A \bullet \arrow[r, "\mathbb{1}_A"] \arrow[rr, bend left, "f \circ \mathbb{1}_A"] & \underset{A}{\bullet} \arrow[r, "f"] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r, "f"] \arrow[rr, bend left, "\mathbb{1}_A \circ f"] & \underset{A}{\bullet} \arrow[r, "\mathbb{1}_A"] & \bullet\ A \end{tikzcd}$$

Figure 1.1: Diagram for Eq. (1.3)

$$\begin{tikzcd} \bullet \arrow[r, "h"] \arrow[rr, bend left, "g \circ h"] \arrow[rrr, bend left=40, "f \circ (g \circ h)"] \arrow[rrr, bend right, "(f \circ g) \circ h"'] & \bullet \arrow[r, "g"] \arrow[rr, bend right, "f \circ g"'] & \bullet \arrow[r, "f"] & \bullet \end{tikzcd}$$

Figure 1.2: Diagram for Eq. (1.2)

Category as a directed graph

So a category is a tuple $\mathscr{C} = (\mathrm{ob}(\mathscr{C}) = V(\mathscr{C}), \mathrm{Mor}(\mathscr{C}), \mathrm{cod}, \mathrm{dom}, \circ, \mathbb{1}_A)$

$$\begin{aligned} \mathrm{cod} : \mathrm{Mor}(\mathscr{C}) &\to \mathrm{ob}(\mathscr{C}) \\ \mathrm{dom} : \mathrm{Mor}(\mathscr{C}) &\to \mathrm{ob}(\mathscr{C}) \\ \mathrm{ob}(\mathscr{C}) &\to \mathrm{Mor}(\mathscr{C}) \\ A &\mapsto \mathbb{1}_A : A \to A \\ \circ : \mathrm{Mor}(\mathscr{C}) \times \mathrm{Mor}(\mathscr{C}) &\to \mathrm{Mor}(\mathscr{C}) \\ (f, g) &\mapsto f \circ g \quad \text{iff } \mathrm{cod}\, g = \mathrm{dom}\, f \end{aligned}$$

$$\begin{tikzcd} \bullet \arrow[r, "f"'] \arrow[rr, bend left, "g \circ f"] & \bullet \arrow[r, "g"'] & \bullet \end{tikzcd}$$

s.t. $(f \circ g) \circ h = f \circ (g \circ h)$ & $\mathbb{1}_A \circ f = f$ & $f \circ \mathbb{1}_A = f$ whenever the compositions are defined.

$$\mathrm{dom} f \bullet \xrightarrow{f \in \mathrm{Mor}(\mathscr{C})} \bullet\ \mathrm{cod} f \qquad f : \mathrm{dom}\, f \longrightarrow \mathrm{cod}\, f$$

Recall: A directed graph is a tuple $E = (E^0, E^1, r, s)$ where $E^0 = v(E)$ is the set of vertices, E^1 is the set of edges, $r : E^1 \to E^0$ is the range map & $s : E^1 \to E^0$ is the source map. So a category is a directed graph with additional structure.

$$s(e) \bullet \xrightarrow{\quad e \quad} \bullet\ r(e) \qquad \begin{tikzcd} \bullet \arrow[r, "e"] \arrow[rr, bend left, "?"] & \bullet \arrow[r, "f"] & \bullet \end{tikzcd}$$

But this has no composition defined on it, since path $\neq$ edge.

If we consider paths instead of edges as morphisms, vertex as path of length 0, edge as path of length 1, then directed graph becomes a category, called the *path category*.

$$\begin{aligned} s(e)e &= e \\ er(e) &= e \end{aligned} \qquad \begin{tikzcd} \underset{v}{\bullet} \arrow[loop left, "g"] \arrow[r, "e"] & \underset{w}{\bullet} \end{tikzcd}$$

$$\begin{tikzcd} \bullet \arrow[r, "e"] \arrow[rr, bend left] & \bullet \arrow[r, "f"] & \bullet \end{tikzcd} \qquad \overline{\quad ef \quad}$$

Example 9

Equivalence class ρ on a set is a category.

$$\overset{(a,b)\in\rho}{a = \mathrm{dom} f \xrightarrow{\quad f \quad} b = \mathrm{cod} f} \qquad \overset{(c,d)\in\rho}{c = \mathrm{dom} g \xrightarrow{\quad g \quad} d = \mathrm{cod} g}$$

For $(a,b)\circ(c,d) = ?$ to make sense we must have $b = c$. So composition is defined as $(a,b)\circ(b,d) = (a,d)$. The identity morphism is (a,a) which exists due to reflexivity. Equivalence class is an example of a directed graph category with inverse arrows. Hence, equivalence class is a groupoid.

Warning. Taking Hasse diagrams in the above example won't work.

Example 10

Suppose $(P, \leqslant)$ is a pre-ordered set, i.e., $\leqslant$ is a pre-order on P (reflexive & transitive). Take

$$\mathrm{ob}(\mathscr{C}) := P,\ \mathrm{Mor}(\mathscr{C}) = \{f : a \to b \mid a \leqslant b\}$$
$$\text{for } a, b \in P,\ f \text{ is unique if exists}$$

so $\mathrm{Mor}(a,b) = \underbrace{\{f\}}_{\text{a singleton}}$ or $\varnothing$. This is a category.
In fact, any small category $\mathscr{C}$ s.t. for any two objects $A, B \in \mathrm{ob}(\mathscr{C})$, $\mathrm{Mor}(A,B)$ is either a singleton or empty, is nothing but a pre-ordered set.
Poset is also a category but due to antisymmetry no inverse arrows exist between distinct objects.

Convention 1. *We don't draw the identity arrow loop, so we draw*

$$A \bullet \qquad \textit{instead of} \qquad \begin{tikzcd} A \bullet \arrow[loop above, "\mathbb{1}_A"] \end{tikzcd}$$

The following are some examples of categories,

Example 11

Discrete category i.e. only morphisms are the identity morphisms

$$\bullet \quad \bullet$$

Example 12

$$\bullet \longrightarrow \bullet$$

Example 13

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd}$$

Example 14

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr] & \bullet \arrow[d] \\ \bullet & \bullet \end{tikzcd}$$

Example 15

$$
\begin{tikzcd}
\bullet \arrow[r] \arrow[d] \arrow[dr] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet
\end{tikzcd}
$$

Example 16

$$
\begin{tikzcd}
\bullet \arrow[r, bend left] \arrow[r, bend right] & \bullet
\end{tikzcd}
$$

The following are *NOT* examples of categories,

Counterexample 2

$$
\begin{tikzcd}
\bullet \arrow[r] & \bullet \\
\bullet \arrow[u] &
\end{tikzcd}
$$

Counterexample 3

$$
\begin{tikzcd}
\bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
\bullet & \bullet
\end{tikzcd}
$$

Counterexample 4

$$
\begin{tikzcd}
\bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet
\end{tikzcd}
$$

The diagram in Example 15 is an example of a *commutative diagram*

$$
\begin{tikzcd}
\bullet \arrow[r, "f"'] \arrow[d, "h"'] \arrow[dr] & \bullet \arrow[d, "g"] \\
\bullet \arrow[r, "u"'] & \bullet
\end{tikzcd}
$$

where the *commutativity* of a diagram of maps, when it makes sense, means that the maps obtained by following the diagram from one initial point to a terminal point along each displayed route are the same. The following diagram is not commutative, as there is another map $v \neq u \circ h = g \circ f$.

$$
\begin{tikzcd}
\bullet \arrow[r, "f"'] \arrow[d, "h"'] \arrow[dr, shift left] \arrow[dr, shift right, "v"'] & \bullet \arrow[d, "g"] \\
\bullet \arrow[r, "u"'] & \bullet
\end{tikzcd}
$$

Derivation of the change-of-basis formula using commutative diagram

Consider two vector spaces V and W over F. Then we have the following commutative diagram

$$\begin{tikzcd}
V \arrow[r, "T"'] \arrow[d, "I_V"'] \arrow[dr, "I_W \circ T = T \circ I_V"] & W \arrow[d, "I_W"] \\
V \arrow[r, "T"'] & W
\end{tikzcd}$$

Let V and W be finite dimensional, consider the bases $\mathcal{B}_1, \mathcal{B}_1'$ of V and $\mathcal{B}_2, \mathcal{B}_2'$ of W.

$$P = [I_V]^{\mathcal{B}_1}_{\mathcal{B}_1'} \text{ (say)} \qquad \begin{tikzcd}
\underset{\mathcal{B}_1'}{\overset{V}{\bullet}} \arrow[r, "T"'] \arrow[d, "I_V"'] \arrow[dr, "I_W \circ T = T \circ I_V"] & \underset{\mathcal{B}_2'}{\overset{W}{\bullet}} \arrow[d, "I_W"'] \arrow[r, dashed] & B = [T]^{\mathcal{B}_2'}_{\mathcal{B}_1'} \text{ (matrix of } T) \\
\overset{\mathcal{B}_1}{\underset{V}{\bullet}} \arrow[r, "T"'] & \overset{\mathcal{B}_2}{\underset{W}{\bullet}} \arrow[r, dashed] & A = [T]^{\mathcal{B}_2}_{\mathcal{B}_1} \text{ (matrix of } T)
\end{tikzcd} \qquad Q = [I_W]^{\mathcal{B}_2}_{\mathcal{B}_2'} \text{ (say)}$$

then $[I_W \circ T]^{\mathcal{B}_2}_{\mathcal{B}_1'} = [T \circ I_V]^{\mathcal{B}_2}_{\mathcal{B}_1'} \implies [I_W]^{\mathcal{B}_2}_{\mathcal{B}_2'} \circ [T]^{\mathcal{B}_2'}_{\mathcal{B}_1'} = [T]^{\mathcal{B}_2}_{\mathcal{B}_1} \circ [I_V]^{\mathcal{B}_1}_{\mathcal{B}_1'}$

$$\implies QB = AP \implies B = Q^{-1}AP \quad \text{(change-of-basis)}$$

Example 17

For any field K, $\mathbf{Vct}_K$ is a category.
$\mathrm{ob}(\mathbf{Vct}_K)$ Objects: all vector spaces over K
$\mathrm{Mor}(\mathbf{Vct}_K)$ Morphisms: all K-linear transformations

1.3 Functor

A *functor* is a correspondence between two categories satisfying some properties. To be precise,

Definition 7: Functor

Let $\mathscr{C}$ & $\mathscr{D}$ be two categories, then $F : \mathscr{C} \longrightarrow \mathscr{D}$ is called a **functor** if it defines two functions[a]

$$\begin{aligned} \mathrm{ob}(\mathscr{C}) &\xrightarrow{F} \mathrm{ob}(\mathscr{D}) \\ A &\longmapsto FA \end{aligned} \quad \& \quad \begin{aligned} \mathrm{Mor}(\mathscr{C}) &\xrightarrow{F} \mathrm{Mor}(\mathscr{D}) \\ f &\longmapsto Ff \end{aligned}$$

s.t.

(i) for any $f \in \mathrm{Mor}(\mathscr{C})$, $\mathrm{dom}(Ff) = F(\mathrm{dom}\, f)$ & $\mathrm{cod}(Ff) = F(\mathrm{cod}\, f)$ (i.e. F commutes with dom and cod)

(ii) for any $A \in \mathrm{ob}(\mathscr{C})$, $F\mathbb{1}_A = \mathbb{1}_{FA}$

(iii) for any $f, g \in \mathrm{Mor}(\mathscr{C})$ if $g \circ f$ is defined in $\mathscr{C}$ then $Fg \circ Ff$ is defined in $\mathscr{D}$ and

$$F(g \circ f) = Fg \circ Ff.$$

[a] there is some abuse of notation as we denote both the functions by the same letter F for convenience

Remark. Sometimes $F(g \circ f) = Ff \circ Fg$.

Remark. For any category $\mathscr{C}$, $\mathbb{1}_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}$ is defined by

$$\begin{aligned} \mathrm{ob}(\mathscr{C}) &\xrightarrow{\mathbb{1}_{\mathscr{C}}} \mathrm{ob}(\mathscr{C}) \\ A &\longmapsto A \end{aligned} \quad \& \quad \begin{aligned} \mathrm{Mor}(\mathscr{C}) &\xrightarrow{\mathbb{1}_{\mathscr{C}}} \mathrm{Mor}(\mathscr{C}) \\ f &\longmapsto \mathbb{1}_{\mathscr{C}} f := f \end{aligned}$$

& it is called the *identity functor* of the category $\mathscr{C}$.

Contravariant and covariant functors

$$\begin{tikzcd} M \arrow[rr, "g"] \arrow[dr, "?"'] & & N_1 \arrow[dl, "f"] \\ & N_2 & \end{tikzcd}$$

Recall: For any two modules M & N, $\mathrm{Hom}(M, N)$ is an abelian group but not a module.
Let $M \in \mathrm{ob}(R-\mathbf{Mod})$. Consider the correspondence $\mathrm{Hom}(M, -)$ between $R-\mathbf{Mod}$ and $\mathbf{Ab}$,

$$R-\mathbf{Mod} \xrightarrow{\mathrm{Hom}(M,-)} \mathbf{Ab}$$

$$\begin{aligned} \mathrm{ob}(R-\mathbf{Mod}) &\xrightarrow{\mathrm{Hom}(M,-)} \mathrm{ob}(\mathbf{Ab}) \\ N &\longmapsto \mathrm{Hom}(M, N) \end{aligned}$$

$$\begin{aligned} \mathrm{Mor}(R-\mathbf{Mod}) &\xrightarrow{\mathrm{Hom}(M,-)} \mathrm{Mor}(\mathbf{Ab}) \\ f &\longmapsto \mathrm{Hom}(M, f) : \mathrm{Hom}(M, N_1) \to \mathrm{Hom}(M, N_2) \\ & \qquad\qquad g \mapsto \mathrm{Hom}(M, f)(g) := ? \end{aligned}$$

Ans. $? = f \circ g$.

Exercise 6. *Prove that* $\mathrm{Hom}(M, -) : R-\mathbf{Mod} \to \mathbf{Ab}$ *as defined above is a functor.*

Let us now consider the correspondence $\mathrm{Hom}(-, M)$ between $R-\mathbf{Mod}$ and $\mathbf{Ab}$,

$$R-\mathbf{Mod} \xrightarrow{\mathrm{Hom}(-,M)} \mathbf{Ab}$$

$$\begin{tikzcd} M & & N_1 \arrow[ll, "g\circ f"'] \arrow[dl, "f"] \\ & N_2 \arrow[ul, "g"] & \end{tikzcd}$$

defined by

$$\begin{aligned} \mathrm{ob}(R-\mathbf{Mod}) &\xrightarrow{\mathrm{Hom}(-,M)} \mathrm{ob}(\mathbf{Ab}) \\ N &\longmapsto \mathrm{Hom}(N, M) \end{aligned}$$

$$\begin{aligned} \mathrm{Mor}(R-\mathbf{Mod}) &\xrightarrow{\mathrm{Hom}(-,M)} \mathrm{Mor}(\mathbf{Ab}) \\ f &\longmapsto \mathrm{Hom}(f, M) : \mathrm{Hom}(N_2, M) \to \mathrm{Hom}(N_1, M) \\ & \qquad\qquad g \mapsto \mathrm{Hom}(f, M)(g) := g \circ f \end{aligned}$$

Notice how $\mathrm{Hom}(-, M)$ reverses the directions of the arrows (*contravariance*) between M & N_1 and M and N_2; whilst $\mathrm{Hom}(M, -)$ keeps the arrow directions same (*covariance*).

Definition 8: Contravariant and covariant functor

If a functor F keeps the directions of the arrows same, i.e. $F(f \circ g) = Ff \circ Fg$, then it is called a **covariant functor**. If, instead, a functor F reverses the directions of the arrows, i.e. $F(f \circ g) = Fg \circ Ff$, then it is called a **contravariant functor**.

Remark. Unless otherwise stated, covariant functor is simply written as functor.

Exercise 7. *Prove that* $\mathrm{Hom}(-, M)$ *is a contravariant functor.*

Example 18

$\mathbb{1}_{\mathbf{Vct}_K} : \mathbf{Vct}_K \to \mathbf{Vct}_K$, the identity functor is covariant.

Question 5. *Is*

$$\begin{aligned} F : \mathbf{Vct}_K &\longrightarrow \mathbf{Vct}_K \\ V &\longmapsto V^{**} \end{aligned} \qquad \textit{a functor ?}$$

Recall: $V^* = \mathcal{L}(V,K)$ is the dual space of V and $V^{**} = (V^*)^* = \mathcal{L}(V^*,K)$ is the double dual space of V. The linear transformation $f : V \to W$

$$V \bullet \xrightarrow{\quad f \quad} \bullet\, W$$

induces another linear transformation $f^* : W^* \to V^*$ in a canonical way, where $\mathcal{B}_1^*, \mathcal{B}_2^*$ are the dual bases of $\mathcal{B}_1, \mathcal{B}_2$.

The map $f^*(g) := g \circ f$ is called the *adjoint* or *transpose* of f. Indeed, $(f \circ g)^* = g^* \circ f^*$ since

$$(f \circ g)^*(h) = h \circ (f \circ g) = (h \circ f) \circ g = (f^*(h)) \circ g = g^*(f^*(h)) = (g^* \circ f^*)(h)$$

whenever the compositions are defined.

$$\begin{tikzcd} \overset{W^*}{\underset{\mathcal{B}_2^*}{\bullet}} \arrow[r] & \overset{V^*}{\underset{\mathcal{B}_1^*}{\bullet}} \\ W \xrightarrow{g} K \arrow[r, mapsto] & V \xrightarrow{f^*(g)} K \end{tikzcd}$$

Figure 1.3: $f^* : W^* \to V^*$

$$\begin{tikzcd} \overset{W}{\underset{\mathcal{B}_2}{\bullet}} \arrow[r, "g"] & \bullet\, K \\ \overset{V}{\underset{\mathcal{B}_1}{\bullet}} \arrow[u, "f"] \arrow[ur, "f^*(g) = g \circ f"'] & \end{tikzcd}$$

Question 6.

$$\begin{tikzcd} V^{**} \bullet \arrow[r, shift left, "Ff\ ?"] & \bullet\, W^{**} \arrow[l, shift left, "?"] \end{tikzcd}$$

What is the definition of Ff ?

By definition $Ff = f^{**} = (f^*)^*$ and

$$\begin{tikzcd} V^{**} \arrow[r, "f^{**}"] & W^{**} \\ h \arrow[r, mapsto] & f^{**}(h) \end{tikzcd} \qquad (f^{**})(h) = (f^*)^*(h) = h \circ f^*$$

so

$$\begin{aligned} f^{**}(h) : W^* &\longrightarrow K \\ g &\longmapsto (f^{**}(h))(g) = (h \circ f^*)(g) = h(f^*(g)) = h \circ (g \circ f) \end{aligned}$$

&

$$\begin{aligned} h : V^* &\longrightarrow K \\ g \circ f &\longmapsto h \circ (g \circ f) \end{aligned}$$

which is well-defined as $f : V \to W$ while $g : W \to K$ so $g \circ f : V \to K$, i.e., $g \circ f \in V^*$.

Also, there is a natural/canonical mapping

$$\begin{aligned} V &\longrightarrow V^{**} \\ v &\longmapsto \hat{v} = \phi_v : V^* \longrightarrow K \\ &\qquad\qquad f \longmapsto \hat{v} = \phi_v(f) := f(v) \end{aligned}$$

which is an example of a *natural transformation*.

1.4 Natural transformation

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Definition 9: Natural transformation

Suppose $\mathscr{C}$ and $\mathscr{D}$ are two categories & $F, G : \mathscr{C} \to \mathscr{D}$ are two functors. A **natural transformation** $\alpha : F \to G$ is defined to be a function from $\text{ob}(\mathscr{C})$ to $\text{Mor}(\mathscr{D})$, denoted by

$$\alpha_A : FA \longrightarrow GA \quad \forall A \in \text{ob}(\mathscr{C})$$

s.t. for any $f : A \to B$ in $\mathscr{C}$ the following diagram[a] commutes

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\alpha_A"'] & FB \arrow[d, "\alpha_B"] \\ GA \arrow[r, "Gf"'] & GB \end{tikzcd} \qquad \boxed{\alpha_B \circ Ff = Gf \circ \alpha_A}.$$

If each α_A is an isomorphism[b] then α is called a **natural isomorphism**.

[a] these diagrams are called *naturality squares* or *naturality diagrams*

[b] i.e. α_A^{-1} exists

Example 19

Let V be a vector space over a field K and V^{**} be its double dual space. We know $V^* = \mathcal{L}(V,K)$, $V^{**} = \mathcal{L}(V^*,K)$. We can define a canonical map $V \longrightarrow V^{**}$

$$\begin{aligned} V &\longrightarrow V^{**} \\ v &\longmapsto \hat{v} = \phi_v : V^* \longmapsto K \\ &\qquad\qquad\qquad f \longmapsto \phi_v(f) := f(v) \end{aligned}$$

which is meaningful as $f \in V^*$ so

$$\begin{aligned} f : V &\longrightarrow K \\ v &\longmapsto f(v) \end{aligned}$$

then

$$\begin{aligned} V &\longrightarrow V^{**} \\ v &\longmapsto \hat{v} \text{ or } \phi_v \end{aligned}$$

is a functor (why ?).

Question 7. *Let $G : \mathbf{Vct}_K \longrightarrow \mathbf{Vct}_K$ s.t.*

$$\begin{aligned} V &\longmapsto V^{**} && \textit{for } \mathrm{ob}(\mathbf{Vct}_K) \\ f &\longmapsto f^{**} && \textit{for } f \in \mathcal{L}(V,W),\ f^{**} \in \mathcal{L}(V^{**},W^{**}) \end{aligned}$$

Is G a functor ?

In this case we get the following naturality square,

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\alpha_V"] & W \arrow[d, "\alpha_W"] \\ V^{**} \arrow[r, "f^{**}"] & W^{**} \end{tikzcd} \tag{1.4}$$

Functor category and Hom functors

Definition 10: Functor category

Let $\mathscr{C}$ and $\mathscr{D}$ be two categories. A **functor category**, denoted $[\mathscr{C},\mathscr{D}]$ or $\mathscr{D}^{\mathscr{C}}$, is a category where $\mathrm{ob}([\mathscr{C},\mathscr{D}])$ is the class of all functors from $\mathscr{C}$ to $\mathscr{D}$ whilst $\mathrm{Mor}([\mathscr{C},\mathscr{D}])$ is the class of all natural transformations between two functors.

Exercise 8. *Show that the functor category $[\mathscr{C},\mathscr{D}]$ above is, indeed, a category.*

Example 20

Let R be a ring. Consider the category $R-\mathbf{Mod}$ of left R-modules and the category $\mathbf{Ab}$ of abelian groups. Let $M \in \mathrm{ob}(R-\mathbf{Mod})$. Define $\mathrm{Hom}(M,-) : R-\mathbf{Mod} \longrightarrow \mathbf{Ab}$ by

$$\begin{aligned} \mathrm{Hom}(M,-) : \mathrm{ob}(R-\mathbf{Mod}) &\longrightarrow \mathrm{ob}(\mathbf{Ab}) \\ N &\longmapsto \mathrm{Hom}(M,N) \end{aligned}$$

$$\begin{aligned} \mathrm{Hom}(M,-) : \mathrm{Mor}(R-\mathbf{Mod}) &\longrightarrow \mathrm{Mor}(\mathbf{Ab}) \\ \mathrm{Hom}(N_1,N_2) \ni f &\longmapsto \mathrm{Hom}(M,f) : \mathrm{Hom}(M,N_1) \longrightarrow \mathrm{Hom}(M,N_2) \\ &\qquad\qquad\qquad\qquad\qquad g \longmapsto \mathrm{Hom}(M,f)(g) := f \circ g \end{aligned}$$

$$\begin{tikzcd} & N_1 \arrow[d, "f"] \\ M \arrow[ur, "g"] \arrow[r, "f\circ g"'] & N_2 \end{tikzcd}$$

Figure 1.4: Composition in Example 20

Exercise 9. *Prove that* $\mathrm{Hom}(M,-)$ *defined above is a functor. This is called a* Hom functor.

Similarly, $\mathrm{Hom}(-, M)$ is a contravariant Hom functor. Hom functors can be generalised to arbitrary categories.

Definition 11: Hom functor

Let $\mathscr{C}$ be any locally small category and $M \in \mathrm{ob}(\mathscr{C})$. Then a Hom **functor** is defined to be a functor $\mathrm{Hom}(M, -) : \mathscr{C} \longrightarrow \mathbf{Set}$ s.t.

$$\begin{aligned} \mathrm{Hom}(M,-) : \mathrm{ob}(\mathscr{C}) &\longrightarrow \mathrm{ob}(\mathbf{Set}) \\ N &\longmapsto \mathrm{Hom}(M,N) \\ \mathrm{Hom}(M,-) : \mathrm{Mor}(\mathscr{C}) &\longrightarrow \mathrm{Mor}(\mathbf{Set}) \\ \mathrm{Hom}(N_1,N_2) \ni f &\longmapsto \mathrm{Hom}(M,f) := f\circ \\ & \qquad \mathrm{Hom}(M,N_1) \longrightarrow \mathrm{Hom}(M,N_2) \end{aligned}$$

and similarly $\mathrm{Hom}(f, M) := \circ f$ for the *contravariant* case.

In this case we get the following naturality square,

$$\begin{tikzcd} \mathrm{Hom}(M,N) \arrow[r, "\mathrm{Hom}(M,f)"] \arrow[d, "\mathrm{Hom}(g,N)"'] & \mathrm{Hom}(M,N') \arrow[d, "\mathrm{Hom}(g,N')"] \\ \mathrm{Hom}(M',N) \arrow[r, "\mathrm{Hom}(M',f)"'] & \mathrm{Hom}(M',N') \end{tikzcd} \tag{1.5}$$

Use of functor

Consider the following sequence of modules and module homomorphisms in the category $R{-}\mathbf{Mod}$,

$$0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0 \tag{1.6}$$

which is said to be *exact* if the the image of each homomorphism is equal to the kernel of the next,

$$\operatorname{im} 0 = \ker f, \ \operatorname{im} f = \ker g, \ \operatorname{im} g = \ker 0.$$

In particular, this sequence is an example of a *short exact sequence*. We have

$$\begin{aligned} \ker f = \operatorname{im} 0 = \{0\} &\implies f \text{ is a monomorphism} \\ \operatorname{im} g = \ker 0 = \{0\} &\implies g \text{ is an epimorphism} \\ M_3 \cong M_2/\operatorname{im} f = M_2/\ker g, &\quad M_2 = M_1 \oplus M_3 \end{aligned}$$

Now apply $\mathrm{Hom}(M, -)$ on the above short exact sequence (Eq. (1.6))

$$\mathrm{Hom}(M,0) \longrightarrow \mathrm{Hom}(M,M_1) \xrightarrow{\mathrm{Hom}(M,f)} \mathrm{Hom}(M,M_2) \xrightarrow{\mathrm{Hom}(M,g)} \mathrm{Hom}(M,M_3) \longrightarrow \mathrm{Hom}(M,0)$$

is this sequence short exact as well ? If yes, the functor $\mathrm{Hom}(M, -)$ is *exact* and the module M is called a *projective module*. The $\mathrm{Hom}(M, -)$ functor is always *left exact*[1], but if it is also *right exact*[2] then it is said to be an *exact functor*.

Hence, we get the very useful characterisation: a module $M \in \mathrm{ob}(R{-}\mathbf{Mod})$ is projective if and only if the functor $\mathrm{Hom}(M, -)$ is exact. Recall: Fix a field K. $\mathbb{1}_{\mathbf{Vct}_K} : \mathbf{Vct}_K \longrightarrow \mathbf{Vct}_K$ is the identity functor in the category of vector spaces over K; further, we have the double dual functor

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$$\begin{aligned} (\ )^{**} : \mathbf{Vct}_K &\longrightarrow \mathbf{Vct}_K \\ V &\longmapsto V^{**} \\ \mathcal{L}(V,W) \ni f &\longmapsto f^{**} \in \mathcal{L}(V^{**}, W^{**}) \end{aligned}$$

Then we have the natural transformation $\alpha : \mathbb{1}_{\mathbf{Vct}_K} \longrightarrow (\ )^{**}$ s.t. for $V \in \mathrm{ob}(\mathbf{Vct}_K)$

$$\begin{aligned} V \longmapsto \alpha_V : \mathbb{1}_{\mathbf{Vct}_K}(V) &\longrightarrow (V)^{**} \\ \alpha_V : V &\longrightarrow V^{**} \\ v &\longmapsto \alpha_V(v) = \hat{v} \end{aligned}$$

[1] i.e. $\mathrm{Hom}(M,0) \to \mathrm{Hom}(M,M_1) \to \mathrm{Hom}(M,M_2) \to \mathrm{Hom}(M,M_3)$ is exact
[2] i.e. $\mathrm{Hom}(M,M_1) \to \mathrm{Hom}(M,M_2) \to \mathrm{Hom}(M,M_3) \to \mathrm{Hom}(M,0)$ is exact

We want to show that the diagram Eq. (1.4) commutes, i.e., the following square

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\alpha_V"] & W \arrow[d, "\alpha_W"] \\ V^{**} \arrow[r, "f^{**}"] & W^{**} \end{tikzcd}$$

is indeed a naturality square; in other words, we have to show that

$$\boxed{f^{**} \circ \alpha_V = \alpha_W \circ f}$$

Let $v \in V$. Then

$$\begin{aligned} (\alpha_W \circ f)(v) &= \alpha_W(f(v)) = \widehat{f(v)} \in W^{**} \\ (f^{**} \circ \alpha_V)(v) &= f^{**}(\alpha_V(v)) = f^{**}(\hat{v}) = \hat{v} \circ f^* \end{aligned}$$

$$\begin{aligned} \widehat{f(v)} : W^* &\longrightarrow K \\ g &\longmapsto \widehat{f(v)}(g) = g(f(v)) \end{aligned}$$

$$\begin{aligned} \hat{v} \circ f^* : W^* &\longrightarrow K \\ g &\longmapsto (\hat{v} \circ f^*)(g) = \hat{v}(f^*(g)) \\ &\qquad = \hat{v}(g \circ f) \\ &\qquad = (g \circ f)(v) \\ &\qquad = g(f(v)) \end{aligned}$$

and as $g \in W^*$ was arbitrary, we have $(\alpha_W \circ f)(v) = \widehat{f(v)} = \hat{v} \circ f^* = (f^{**} \circ \alpha_V)(v)$ for all $v \in V$.

Hence the diagram commutes and $\alpha : \mathbb{1} \longrightarrow (\ )^{**}$ is a natural transformation. This is also why

$$\begin{aligned} \alpha_V : V &\longrightarrow V^{**} \\ v &\longmapsto \hat{v} \end{aligned}$$

is called a *natural/canonical map*.

1.5 Equivalent and isomorphic categories, opposite category

Definition 12: Equivalent and isomorphic categories

Let $\mathscr{C}$ and $\mathscr{D}$ be two categories and

$$F : \mathscr{C} \to \mathscr{D}, \qquad G : \mathscr{D} \to \mathscr{C}$$

be two functors. If there is a natural isomorphism

$$\eta : \mathbb{1}_{\mathscr{C}} \to GF$$

& a natural isomorphism

$$\zeta : FG \to \mathbb{1}_{\mathscr{D}}$$

then we write $\mathscr{C} \cong \mathscr{D}$ (or $\mathscr{C} \simeq \mathscr{D}$) and say that $\mathscr{C}$ and $\mathscr{D}$ are **equivalent categories**.
If $GF = \mathbb{1}_{\mathscr{C}}$ and $FG = \mathbb{1}_{\mathscr{D}}$ then $\mathscr{C}$ and $\mathscr{D}$ are said to be **isomorphic categories**.

Definition 13: Opposite category

If $\mathscr{C}$ is a category, then $\mathscr{C}^{\mathrm{op}}$ with $\mathrm{ob}(\mathscr{C}^{\mathrm{op}}) = \mathrm{ob}(\mathscr{C})$ &

$$\mathrm{Mor}_{\mathscr{C}^{\mathrm{op}}}(A, B) = \mathrm{Mor}_{\mathscr{C}}(B, A)$$

is called the **opposite category** of $\mathscr{C}$.

1.6 Adjunction

Definition 14: Left adjoint

Let $\mathscr{C}$ & $\mathscr{D}$ be two categories and

$$F : \mathscr{C} \to \mathscr{D}, \qquad G : \mathscr{D} \to \mathscr{C}$$

be two functors. Then F is said to be a **left adjoint** of G, written as $F \dashv G$, if for any $A \in \mathrm{ob}(\mathscr{C})$ & for any $B \in \mathrm{ob}(\mathscr{D})$ there is a bijection between

$$\mathrm{Mor}_{\mathscr{D}}(FA, B) \quad \& \quad \mathrm{Mor}_{\mathscr{C}}(A, GB).$$

Note. When F is a left adjoint of G, G is a right adjoint of F.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] & \mathcal{D} \\
A \arrow[r, mapsto] & FA \\
\mathcal{C} \arrow[r, "G"] & \mathcal{D} \\
B \arrow[r, mapsto] & GB
\end{tikzcd}$$

Question 8. *Suppose $\mathscr{C}$ and $\mathscr{D}$ are equivalent categories,*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, shift left, "F"] & \mathcal{D} \arrow[l, shift left, "\simeq G"] \\
A & B
\end{tikzcd} \quad \textit{then do we have} \quad \mathrm{Mor}(FA, B) \xleftrightarrow{?} \mathrm{Mor}(A, GB)$$

i.e. is there a bijection between $\mathrm{Mor}(FA, B)$ *and* $\mathrm{Mor}(A, GB)$ *?*

From the definition of equivalent categories, there are two natural isomorphisms

$$\alpha : \mathbb{1}_{\mathscr{C}} \longrightarrow GF, \quad \beta : FG \longrightarrow \mathbb{1}_{\mathscr{D}}$$
$$\alpha_A : A \longrightarrow GFA, \quad \beta_B : FGB \longrightarrow B$$

where $A \in \mathrm{ob}(\mathscr{C})$ and $B \in \mathrm{ob}(\mathscr{D})$.

$$\begin{tikzcd}
\mathrm{Mor}(FA, B) & \mathrm{Mor}(A, GB) \\
f \arrow[r, mapsto, "\psi"] & Gf \circ \alpha_A \\
\beta_B \circ Fh & h \arrow[l, mapsto, "\phi"]
\end{tikzcd}$$

We first show injectivity. Let $\psi(f) = Gf \circ \alpha_A = Gg \circ \alpha_A = \psi(g)$. Then using "right cancellation" we get

$$\begin{aligned} Gf &= Gg \\ FGf &= FGg \end{aligned} \tag{1.7}$$

We then have the following diagram

$$\begin{tikzcd}
FGFA \arrow[r, "\substack{FGg \\ FGf}"] \arrow[d, "\beta_{FA}"'] & FGB \arrow[d, "\beta_B"] \\
FA \arrow[r, "\substack{f \\ g}"'] & B
\end{tikzcd}$$

whence we get

$$\begin{aligned} \beta_B \circ FGf &= f \circ \beta_{FA} \\ \beta_B \circ FGg &= g \circ \beta_{FA} \end{aligned} \tag{1.8}$$

then, since $\beta_B \circ FGf = \beta_B \circ FGg$, we get

$$\begin{aligned} &f \circ \beta_{FA} = g \circ \beta_{FA} \\ &\implies f = g \end{aligned}$$

where we "right cancel" once again. Hence ψ is injective. For surjectivity, take an $h \in \mathrm{Mor}(A, GB)$

$$A \xrightarrow{h} GB$$

$$\begin{tikzcd} FA \arrow[r, "Fh"] \arrow[d, "?"'] & FGB \arrow[dl, "\beta_B"] \\ B & \end{tikzcd}$$

Ans. $? = \beta_B \circ Fh$.

$$\begin{tikzcd} GFA \arrow[r, "Gf"] & GB \\ A \arrow[u, "\alpha_A"] \arrow[ur, "Gf \circ \alpha_A"'] & \end{tikzcd}$$

$$\begin{aligned} &(\phi \circ \psi)(f) \\ &= \phi(Gf \circ \alpha_A) \\ &= \beta_B \circ (F(Gf \circ \alpha_A)) \\ &= \beta_B \circ (FGf \circ F\alpha_A) \end{aligned}$$

$$\begin{tikzcd} FA \arrow[r, "F(Gf \circ \alpha_A)"] \arrow[dr, "{\beta_B \circ (F(Gf \circ \alpha_A)) = (\phi \circ \psi)(f) \stackrel{?}{=} f}"'] & FGB \arrow[d, "\beta_B"] \\ & B \end{tikzcd}$$

Using Eq. (1.8), $\beta_B \circ (FGf \circ F\alpha_A) = (\beta_B \circ FGf) \circ F\alpha_A = (f \circ \beta_{FA}) \circ F\alpha_A = f \circ (\beta_{FA} \circ F\alpha_A)$

2. Unitary Modules

We will assume all rings to have identity unless otherwise stated. Recall the definition of a unitary left R-module: given $M \neq \varnothing$, $(M,+)$ an abelian group, $R,+,\cdot$ a ring and a binary operation

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$$\begin{aligned} \cdot : R \times M &\longrightarrow M \\ (r,m) &\longmapsto rm \end{aligned}$$

satisfying the following properties:

(i) $r(m_1 + m_2) = rm_1 + rm_2$

(ii) $(r_1 + r_2)m = r_1 m + r_2 m$

(iii) $(r_1 r_2)m = r_1(r_2 m)$

(iv) $1_R m = m$ (unitary module)

for $r, r_1, r_2 \in R$ and $m, m_1, m_2 \in M$; then M is said to be a unitary left R-module. We will assume all modules to be unitary unless otherwise stated. Note that left and right R-modules are not the same unless R is commutative. If R is not commutative, we will by default use R-module to mean *left* R-module.

Example 21

(i) Every vector space over a field F is an F-module.

(ii) Every additive abelian group is a $\mathbb{Z}$-module in *exactly one way* as we've seen.

(iii) Every ring R itself is an R-module.

2.1 Submodules

A lot of the theory for vector spaces and modules is similar, until we get to anything that requires commutativity or division by nonzero elements. Exactly analogous to the notion of a subspace of a vector space, we have the notion of a submodule:

Definition 15: Submodule

A nonempty subset $S \subseteq M$ of an R-module M is a **submodule** of M if S itself is an R-module. Equivalently, $\varnothing \neq S \subseteq M$ is a submodule of M if for all $r \in R$ and $x, y \in S$ we have $x + ry \in S$.

Example 22

For any ring R the submodules of R considered as a module over itself are the ideals I of R. For example, all the submodules of $\mathbb{Z}$ are $n\mathbb{Z}$.

Notation 1. *We write $S \leqslant M$ to mean that S is a submodule of M.*

Recall that the sum and intersections of subspaces is also a subspace, but the union of subspaces need not be a subspace. We have similar results for submodules:

Theorem 2

Let M be an R-module and $S_1, S_2 \leqslant M$. Define $S_1 + S_2 = \{s_1 + s_2 : s_1 \in S_1, s_2 \in S_2\}$. Then $S_1 + S_2, S_1 \cap S_2 \leqslant M$ but $S_1 \cup S_2 \not\leqslant M$.

Proof. As $S_1, S_2 \leqslant M$, clearly $0 \in S_1, S_2$ so $0 = 0 + 0 \in S_1 + S_2$ and $0 \in S_1 \cap S_2$. Let $r \in R$. Then

$$s_1 + s_2, s_1' + s_2' \in S_1 + S_2 \implies s_1 + s_2 + r(s_1' + s_2') = (s_1 + rs_1') + (s_2 + rs_2') \in S_1 + S_2$$

as $s_1 + rs_1' \in S_1$ and $s_2 + rs_2' \in S_2$ due to $S_1, S_2 \leqslant M$; also,

$$s, s' \in S_1 \cap S_2 \implies s, s' \in S_1, S_2 \implies s + rs' \in S_1, S_2 \implies s + rs' \in S_1 \cap S_2.$$

Hence, $S_1 + S_2, S_1 \cap S_2 \leqslant M$. Now, consider the $\mathbb{Z}$-module $\mathbb{Z}$. Then $2\mathbb{Z}, 3\mathbb{Z} \leqslant \mathbb{Z}$ but $2\mathbb{Z} \cup 3\mathbb{Z} \not\leqslant \mathbb{Z}$; take $2 \in 2\mathbb{Z}$, $3 \in 3\mathbb{Z}$ then $2 + 3 = 5 \notin 2\mathbb{Z} \cup 3\mathbb{Z}$. ■

Every vector space has a direct sum decomposition: both trivial ($U \oplus \{0\} = V$) and non-trivial ($U \oplus U^{\perp} = V$.) However, the same is not true for modules. Consider again the $\mathbb{Z}$-module $\mathbb{Z}$ and if possible let $\mathbb{Z} = n\mathbb{Z} \oplus m\mathbb{Z}$. For any two algebraic structures A, B we have $A \oplus B \cong A \times B$, by the map

$$\begin{aligned} f : & A \oplus B \longrightarrow A \times B \\ & a + b \longmapsto (a, b) \end{aligned}$$

whence, taking $G = A \oplus B \cong A \times B$, one gets $G/A \cong B$. So from our assumption,

$$\mathbb{Z} = n\mathbb{Z} \oplus m\mathbb{Z} \cong n\mathbb{Z} \times m\mathbb{Z} \implies \mathbb{Z}/n\mathbb{Z} \cong m\mathbb{Z}$$

which is absurd as $\mathbb{Z}/n\mathbb{Z}$ is finite so it cannot be isomorphic to the infinite $m\mathbb{Z}$.

2.2 Free modules

The following definitions are analogous to that of vector spaces or even groups,

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Definition 16: Submodules generated by a set

Let M be an R-module and $S \neq \varnothing$ be any nonempty set. The **submodule $\langle S \rangle$ generated by S**[a] is the smallest submodule of M containing S, i.e.,

$$\operatorname{span}(S) = \langle S \rangle = \bigcap_{S \subseteq N_S \leqslant M} N_S.$$

If $S = \{x_1, \ldots, x_n\}$ is finite, then we write $\langle x_1, \ldots, x_n \rangle = \langle \{x_1, \ldots, x_n\} \rangle$ and call the submodule *finitely generated*. In particular,

$$\langle x_1, \ldots, x_n \rangle = Rx_1 + \cdots + Rx_n.$$

For $n = 1$, $\langle x_1 \rangle = Rx_1$ is called the *cyclic submodule* generated by x_1.

[a] or, from vector space terminology, the *span* $\operatorname{span}(S)$ of S

Definition 17: Linear independence

A nonempty subset S of an R-module M is said to be **R-linearly independent** (or simply *linearly independent*) if for all possible finite subsets $\{x_1, \ldots, x_n\}$ of S we have

$$r_1 x_1 + \cdots r_n x_n = 0 \implies r_1 = \cdots = r_n = 0 \quad \forall r_1, \ldots, r_n \in R.$$

Definition 18: Basis

A subset $\mathcal{B}$ of an R-module M is an **R-basis** (or simply *basis*) of M if $\mathcal{B}$ is *linearly independent* and $\operatorname{span} \mathcal{B} = \langle \mathcal{B} \rangle = M$.

Recall that for a vector space V over a field F the following are equivalent:

(1) $\mathcal{B}$ is an F-basis.

(2) $\mathcal{B}$ is a maximal linearly independent subset of V.

(3) $\mathcal{B}$ is a minimal spanning set of V.

Is the basis ever unique for a (non-trivial) vector space ? A group G has a unique generator iff $|G| = 2$. Similarly, V has a unique F-basis iff $\dim_F(V) = 1, |F| = 2$. For a vector space (group) we can always find a basis (set of generators), and the same is true for its subspaces (subgroups). However, the same is not true for modules - and even if a basis $\mathcal{B}$ *does exist*, we can sometimes find another basis $\mathcal{B}'$ s.t. $|\mathcal{B}| \neq |\mathcal{B}'|$!

The $\mathbb{Z}$-module $\mathbb{Z}$ has the bases $\{1\}$ and $\{-1\}$ but the $\mathbb{Z}$-modules $\mathbb{Q}$ and $\mathbb{Z}/(n)$ don't have any basis. We say a module M is a *free module* if it has a basis. Any ring R over itself is a free module. If for every finitely generated module over R, the bases have the same cardinality, then R is said to have the *invariant basis number (IBN)* property.

$\mathbb{Z} \times \mathbb{Z}$ is a free module over itself but $\mathbb{Z} \times \{0\}$ is not a free module over $\mathbb{Z} \times \mathbb{Z}$.

Theorem 3: Modular law

If M is an R-module and $A, B, C \leqslant M$ with $C \subseteq A$, then

$$A \cap (B + C) = (A \cap B) + C.$$

Proof. $C \subseteq A$ so $A \cap C = C$. Also, $A \cap B \subseteq A, B$. Now,

$$(A \cap B) + C \subseteq A + C \subseteq A + A \subseteq A,$$
$$(A \cap B) + C \subseteq A + C \subseteq B + C.$$

Hence, $(A \cap B) + C \subseteq A \cap (B + C)$. Now let $x \in A \cap (B + C)$. Then $x \in A$ and $x \in B + C$,

$$\implies x = b + c,\ b \in B, c \in C \implies x - c = b \in B.$$

But $c \in C \subseteq A$, so $x - c \in A$. Hence, $x = x - c + c \in (A \cap B) + C$. ∎

Exercise 10. *(1) Show that the $\mathbb{Z}$-module $\mathbb{Q}$ has no finite set of generators.*

(2) Show that the $\mathbb{Z}$-module $\mathbb{Q}$ cannot be expressed as a direct sum of two nonzero submodules of $\mathbb{Q}$.

(3) If e is a central idempotent *in a ring R and M is an R-module, then show that*

$$M = eM \oplus (1 - e)M.$$

Solution. (1) Suppose for contradiction that there is a finite set of generators $\left\{\frac{p_1}{q_1}, \ldots, \frac{p_n}{q_n}\right\}$ with $\gcd(p_i, q_i) = 1$, $q_i \neq 0$. Then

$$r_1 \frac{p_1}{q_1} + \cdots + r_n \frac{p_n}{q_n} = \frac{p}{q} \quad \forall \frac{p}{q} \in \mathbb{Q}.$$

Now there exists a prime p s.t. p does not divide $q_1, \ldots, q_n$; clearly, as there are infinitely many primes. Let

$$r_1 \frac{p_1}{q_1} + \cdots + r_n \frac{p_n}{q_n} = \frac{p}{q} \quad \forall \frac{1}{p},$$

then

$$r_1 p_1 + \cdots + r_n p_n = \frac{q_1 \cdots q_n}{p}$$

where the LHS is an integer, so the RHS must be as well but then p divides $q_1 \cdots q_n$ so p divides at least one of $q_1, \ldots, q_n$; contradiction.

(2) Let $\mathbb{Q} = \mathbb{Q}_1 \oplus \mathbb{Q}_2$ then $\mathbb{Q} = \mathbb{Q}_1 + \mathbb{Q}_2$ with $\mathbb{Q}_1 \cap \mathbb{Q}_2 = \{0\}$. Let

$$q_1 = \frac{a_1}{b_1} \in A \setminus \{0\},\ q_2 = \frac{a_2}{b_2} \in B \setminus \{0\}$$

then $b_1 a_2 q_1 \in A$ and $b_2 a_1 q_2 \in B$. Now,

$$b_1 a_2 q_1 = b_1 a_2 \frac{a_1}{b_1} = a_2 a_1,$$

$$b_2 a_1 q_2 = b_2 a_1 \frac{a_2}{b_2} = a_1 a_2$$

so $b_1 a_2 q_1 = a_1 a_2 = b_2 a_1 q_2 \in A \cap B$. Now, $a_1, a_2 \neq 0$ but $A \cap B = \{0\}$; contradiction.

(3) Let e be a central idempotent i.e. $e^2 = e$ and $er = re$ for all $r \in R$. Let M be an R-module, then

$$eM = \{em : m \in M\}, \quad (1 - e)M = \{(1 - e)m : m \in M\}.$$

It is easy to see that $eM, (1 - e)M \leqslant M$. Clearly, $M = eM + (1 - e)M$ as $m = em + (1 - e)m \in eM + (1 - e)M$ for any $m \in M$. It remains to show that the sum is direct. Clearly $\{0\} \subseteq eM \cap (1 - e)M$.

Now let $m \in eM \cap (1-e)M$, then $m = em_1$ and $m = (1-e)m_2$ for some $m_1, m_2 \in M$. So,

$$em = e(1-e)m_2 = (e - e^2)m_2 = (e-e)m_2 = 0m_2 = 0$$

so $em = 0$. Further, $m = em_1 \implies em = e^2m_1 = em_1 \implies em_1 = em = 0$. But then $m = em_1 = 0$, so $eM \cap (1-e)M = \{0\}$ and hence the sum is direct. ∎

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2.3 Quotient modules and isomorphisms

For an algebraic structure, it makes sense to ask whether we can study mappings on that structure that preserve said structure i.e. *homomorphisms*. The study of homomorphisms naturally leads to *quotient* structures and *isomorphic* structures. We have already done this for groups, rings and vector spaces. We simply generalise those familiar notions to that of a module.

A **module homomorphism** is simply a map f that preserves the module properties:

$$f(a + rb) = f(a) + rf(b)\ \forall a, b \in M,\ r \in R.$$

Further, f is a *monomorphism* if injective, an *epimorphism* if surjective, and an *isomorphism* if both i.e. bijective. The *image* of f is

$$\operatorname{im} f = f(M) = \{f(m) : m \in M\}$$

and the *kernel* of f is

$$\ker f = \{m \in M : f(m) = 0\}.$$

Then $\ker f \leqslant M$ and $\operatorname{im} f \leqslant N$.

Definition 19: Quotient module

Given a ring R, module M and submodule $N \leqslant M$, we define the **quotient module** M/N as

$$M/N = \{a + N : a \in M\}$$

s.t.

(1) $(a+N) + (b+N) = (a+b) + N$, and

(2) $(a+N)(b+N) = ab + N$.

A quotient module always exists: the natural example is $M/\ker f$ for any homomorphism f on M. In particular, if f is an epimorphism then we know that $M/\ker f \cong \operatorname{im} f$.

Theorem 4: Isomorphism theorems

Let R be a ring and M a module.

(1) $M/\ker f \cong N$ if $f : M \to N$ is an epimorphism.

(2) $N_1, N_2 \leqslant M$ then $N_1/(N_1 \cap N_2) \cong (N_1 + N_2)/N_2$.

(3) $A, B \leqslant M$ s.t. $A \subseteq B$ then $(M/A)/(B/A) \cong M/B$.

2.4 Simple modules

Simple group: $G \neq \{e\}$ s.t.

$$H \trianglelefteq G \implies H = \{e\},\ G.$$

Simple ring: $R^2 \neq \{0\}$ s.t.

$$I \trianglelefteq R \implies I = \{0\},\ R.$$

Every simple abelian group is isomorphic to $\mathbb{Z}/p\mathbb{Z}$. Every commutative simple ring is a field.

Definition 20: Simple module

$M \neq \{0\}$ is said to be a **simple module** if the only submodules of M are $\{0\}$ and M itself.

Proposition 1

Every commutative simple $\mathbb{Z}$-module is isomorphic to $\mathbb{Z}/p\mathbb{Z}$.

Lemma 1: Schur's Lemma

If M is a simple R-module then $\operatorname{End}_R(M)$ is a division ring.

Proof. Let $f \in \operatorname{End}_R(M)$ s.t. $f \neq 0$. Then $\operatorname{im} f, \ker f \leqslant M$. As M is simple, this implies $\operatorname{im} f = \{0\}$ or M, $\ker f = \{0\}$ or M. If $\ker f = \{0\}$ (injective) and $\operatorname{im} f = M$ (surjective) then we're done. If $\operatorname{im} f = \{0\}$ then f maps every element to 0 i.e. $f = 0$ which is absurd. If $\ker f = M$, then by the same argument $f = 0$ which is absurd. ∎

Definition 21: Semisimple module

M is said to be a **semisimple module** (or *completely reducible*) if it is the direct sum of *simple* (*irreducible*) modules.

Definition 22: Regular ring [Neu36, von Neumann, 1936]

A ring R is said to be a **von Neumann regular ring** or simply a **regular ring** if for all $a \in R$ there exists an $x \in R$ s.t. $\boxed{axa = a}$.

Corollary 1: Corollary to Schur's Lemma 1

If M is a semisimple R-module then $\mathrm{End}_R(M)$ is a regular ring.

Exercise 11. 1. *Let M be an R module and $f \in \mathrm{End}_R(M)$ s.t. $f^2 = f$. Show that*

$$\boxed{\ker f \oplus \operatorname{im} f = M}.$$

2. *Let R be an integral domain. Given $x \in R$ with $x \neq 0$, show that there is a descending chain*

$$R \supseteq Rx \supseteq Rx^2 \supseteq \cdots \supseteq Rx^n \supseteq Rx^{n+1} \supseteq \cdots$$

of submodules of the R-module R s.t. for every n, $Rx^n / Rx^{n+1} \cong R/Rx$.

3. *Show that every finitely generated R-module is a homomorphic image of a finitely generated free R-module.*

4. *If M is a cyclic R-module, then show that $M \cong R/I$, where I is a left ideal of R. If M is simple, then show that $M \cong R/\mathfrak{m}$, where $\mathfrak{m}$ is a maximal left ideal of R.*

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