

MODULE THEORY

Semester 7, 2026

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$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\ 0 \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \end{array}$$

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1. Modules & Categories

1.1 Modules

A vector space is a field action whereas a left (right) module is a left (right) ring action. A field is commutative so left and right vector spaces are the same.

LECTURE 1
21st Jul, 2026

Definition 1: Vector space

Let F be a field and V be a nonempty set. Then V is a **vector space** over F if

- (i) there exists an addition $+$ on V s.t. $(V, +)$ is an abelian group; &
- (ii) there exists a mapping^a

$$\begin{aligned} F \times V &\longrightarrow V \\ (\lambda, v) &\longmapsto \lambda v \end{aligned}$$

satisfying

- (a) $\lambda(v + w) = \lambda v + \lambda w$ for all $\lambda \in F$ and $v, w \in V$;
- (b) $(\lambda + \mu)v = \lambda v + \mu v$ for all $\lambda, \mu \in F$ and $v \in V$;
- (c) $(\lambda\mu)v = \lambda(\mu v)$ for all $\lambda, \mu \in F$ and $v \in V$; and
- (d) $1_F v = v$ for all $v \in V$, where 1_F is the multiplicative identity of F .

^afield action

Definition 2: Left module

Let R be a ring and M be a nonempty set. Then M is a **left module** over R , or a **left- R module** if

- (i) there exists an addition $+$ on M s.t. $(M, +)$ is an abelian group; &
- (ii) there exists a mapping^a

$$\begin{aligned} R \times M &\longrightarrow M \\ (r, m) &\longmapsto rm \end{aligned}$$

satisfying

- (a) $r(m + n) = rm + rn$ for all $r \in R$ and $m, n \in M$;
- (b) $(r + s)m = rm + sm$ for all $r, s \in R$ and $m \in M$;
- (c) $(rs)m = r(sm)$ for all $r, s \in R$ and $m \in M$; and

^aleft ring action

Similarly, one can define a right R -module by considering the right action $M \times R \rightarrow M$ instead.

Note. For a left R -module M , in axiom (c), $(rs)m = r(sm)$ need not be equal to $s(rm)$, $r, s \in R, m \in M$. For a right R -module M ,

$$m(rs) = (mr)s \neq (ms)r.$$

Differences between vector space and module

- (i) Unless the ring is commutative (for a module), the left action (for a module) and the right action are different. But for a vector space over a field, the left action gives rise to the right action and vice versa.
- (ii) A ring may not have an identity and in that case there is no meaning of the analogous defining condition (d) of Definition 1 for a module. If the ring R has an identity 1_R , and $1_R m = m$ for all $m \in M$, then M is called a **unitary** or **unital left R -module**.

If the ring is commutative, then the notion of a left and right R -module is the same. In this case, we simply call it an R -module. In fact, whenever it is clear, we will say R -module to mean *left* R -module if R is not necessarily commutative.

Suppose G is an (additive) abelian group. Recall that for any $n \in \mathbb{Z}$ and $a \in G$,

$$na := \begin{cases} a + a + \cdots + a & (n \text{ times}), & \text{if } n > 0 \\ (-a) + (-a) + \cdots + (-a) & (|n| \text{ times}), & \text{if } n < 0 \\ 0_G, & & \text{if } n = 0. \end{cases}$$

The above formula for na is in fact an action of $\mathbb{Z}$ on G , i.e.,

$$\begin{aligned} \mathbb{Z} \times G &\longrightarrow G \\ (n, a) &\longmapsto na \end{aligned}$$

such that the following properties are satisfied

- (i) $n(a + b) = na + nb$ for all $n \in \mathbb{Z}$ and $a, b \in G$;
- (ii) $(m + n)a = ma + na$ for all $m, n \in \mathbb{Z}$ and $a \in G$; and
- (iii) $(mn)a = m(na)$ for all $m, n \in \mathbb{Z}$ and $a \in G$.
- (iv) $1a = a$ for all $a \in G$.

Thus, G is a (unitary) $\mathbb{Z}$ -module.



Note. Thus any (additive) abelian group is canonically a $\mathbb{Z}$ -module. Conversely, any $\mathbb{Z}$ -module is an (additive) abelian group.



Remark. One can see a module as a generalisation of an (additive) abelian group (just by replacing the natural $\mathbb{Z}$ -action by an R -action for an arbitrary ring R) or as a generalisation of a vector space over a field (by replacing the action of a field F by the action of an arbitrary ring R).

Ring of endomorphisms of an abelian group

Question 1. Suppose $(M, +)$ is an abelian group and R is a ring. Is there any guarantee that R has an action (left or right) on M s.t. M becomes an R -module?

To answer the above question it may help to consider the following question:

Question 2. What is the difference or connection between a group action and a ring action (giving rise to a module)? OR Is there any analogy between the two?

Recall: If a group has an action on a set S then there exists a group homomorphism

$$\begin{aligned} G &\longrightarrow \text{Sym}(S) \\ a &\longmapsto \phi_a : S \rightarrow S \\ &\quad x \mapsto ax \end{aligned}$$

induced by the action of G on S (often denoted $G \curvearrowright S$).

Conversely, if there exists a group homomorphism f from G to the canonical group $\text{Sym}(S)$

$$\begin{aligned} f : G &\longrightarrow \text{Sym}(S) \\ g &\longmapsto f(g) : S \rightarrow S \end{aligned}$$

then

$$\begin{aligned} G \times S &\longrightarrow \text{Sym}(S) \\ (g, s) &\longmapsto gs := f(g)(s) \end{aligned}$$

defines an action of G on S . G always has an action on S , the trivial action

$$\begin{aligned} G \times S &\rightarrow S \\ (g, s) &\mapsto gs = s \end{aligned}$$

Similar to how $\text{Sym}(S)$ is the *canonical group*, we want to find an analogous *canonical ring*.

Recall: Suppose $(M, +)$ is an abelian group. Then

$\text{End}(M) = \text{Hom}(M, M) =$ the set of all group endomorphisms of M

is a ring, where $+$ and $\cdot$ are defined as follows: for all $f, g \in \text{End}(M)$,

$$\begin{aligned} f + g : M &\rightarrow M \\ x &\mapsto (f + g)(x) \\ (f + g)(x) &:= f(x) + g(x) \quad \forall x \in M \end{aligned}$$

and

$$\begin{aligned} fg : M &\rightarrow M \\ x &\mapsto (fg)(x) \\ (fg)(x) &:= f(g(x)) \quad \forall x \in M \end{aligned}$$

i.e. $fg = f \circ g$.

Exercise 1. Prove that $f + g, fg \in \text{End}(M)$. Hence, prove that $(\text{End}(M), +, \cdot)$ is a ring.

This ring $\text{End}(M)$ is the *canonical ring*.

Exercise 2. Suppose M is a (left) R -module. Prove that $f : R \rightarrow \text{End}(M)$ defined by

$$\begin{aligned} f(r) : M &\rightarrow M \\ m &\mapsto f(r)(m) := rm \end{aligned}$$

for all $r \in R, m \in M$; is a ring homomorphism.

Exercise 3. Suppose M is an (additive) abelian group & R is a ring s.t. $\phi : R \rightarrow \text{End}(M)$ is a ring homomorphism. Prove that with this action of R on M , M becomes a left R -module.

The trivial action

$$\begin{aligned} R \times M &\rightarrow M \\ (r, m) &\mapsto rm = 0 \end{aligned}$$

satisfies all the defining conditions of a left R -module but not the defining condition of a unitary module.

So, just as group action is representation of a group by permutation, module is representation of a ring as an endomorphism. To summarise: a group action $G \curvearrowright S$ induces a group homomorphism

$$G \curvearrowright S \mapsto \phi \in \text{Hom}(G, \text{Sym}(S))$$

where $\phi(g)(s) = \phi_g(s) = gs$; but what about the converse, i.e.

$$G \curvearrowright S \xleftarrow{?} \psi \in \text{Hom}(G, \text{Sym}(S))$$

The answer is yes (take $\psi = \phi$) and so $G \curvearrowright S \longleftrightarrow \phi \in \text{Hom}(G, \text{Sym}(S))$. Similar to how $\text{Sym}(S)$ is the canonical group, $\text{End}(M, +, \cdot)$ is the canonical ring where $(M, +)$ is an abelian group and $\text{End}(M) = \text{Hom}(M, M)$ is the set of all group endomorphisms on M . Then

$$\begin{aligned} \psi : R &\longrightarrow \text{End}(M) \\ r &\longmapsto \psi(r) : M \longrightarrow M \\ m &\longmapsto rm \end{aligned}$$

is a ring homomorphism.

Group representations

- (1) Permutation representation $G \curvearrowright \text{Sym}(S)$;
- (2) Linear representation $G \curvearrowright \text{GL}(V)$, where $\text{GL}(V)$ is the set of all bijective linear operators on V (endomorphisms on V); and
- (3) Matrix representation $G \curvearrowright \text{GL}_n(F)$, where F is a field.

LECTURE 2
24th Jul, 2026

The permutation representations give rise to linear representations, and thus also matrix representations, of a group. Suppose $G \curvearrowright \text{Sym}(S)$. Then there exists a vector space V over a field F s.t. any basis $\mathcal{B}$ of V has the property that $|\mathcal{B}| = |S|$.

Suppose $\mathcal{B} = \{v_i\}_{i \in S}$. Let $\sigma \in \text{Sym}(S)$. Then there exists (why ?) a linear operator

$$\begin{aligned} T_\sigma : V &\longrightarrow V \\ v_i &\longmapsto v_{\sigma(i)} \end{aligned}$$

then $\{T_\sigma(v_i)\}_{i \in S} = \{v_{\sigma(i)}\}_{i \in S}$ is a basis of V . Since T_σ maps the basis $\{v_i\}_{i \in S}$ to the basis $\{v_{\sigma(i)}\}_{i \in S}$, therefore $T_\sigma \in \text{GL}(V)$. Hence, we get a correspondence

$$\begin{aligned} G &\xrightarrow{\psi} \text{Sym}(S) \longrightarrow \text{GL}(V) \\ g &\longmapsto \psi_g = \sigma \longmapsto T_\sigma \end{aligned}$$

Here $\dim V = |G|$. Recall how we proved Cayley's theorem:

Theorem 1: Cayley

Every group G is isomorphic to a subgroup of the symmetric group $\text{Sym}(G)$.

Proof. We consider the action of G on itself,

$$\begin{aligned} G &\longrightarrow \text{Sym}(G) \\ g &\longmapsto \lambda_g : G \longrightarrow G \\ a &\longmapsto ga \end{aligned} \tag{1.1}$$

which induces a group homomorphism $\phi(g) = \lambda_g$ on G . Let $g \in \ker \phi$, then $g = ge = \lambda_g(e) = e$ so $\ker \phi = \{e\}$ and the action is faithful. Thus, by the first isomorphism theorem, $\text{im } \phi \cong G / \ker \phi = G / \{e\} = G$. ■

The permutation representation $\phi(g) = \lambda_g$ above in Eq. (1.1) is called a *regular representation*. Take the Klein 4-group $V_4 = \{e, a, b, c\}$ defined by the Cayley table

$\cdot$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

and consider the left multiplication actions by each element,

$$\begin{aligned} \lambda_e : e &\longmapsto e \\ a &\longmapsto ea = a \\ b &\longmapsto eb = b \\ c &\longmapsto ec = c \\ \lambda_a : e &\longmapsto a \\ a &\longmapsto e \\ b &\longmapsto ab = c \\ c &\longmapsto ac = b \\ \lambda_b : e &\longmapsto b \\ a &\longmapsto ba = c \\ b &\longmapsto e \\ c &\longmapsto bc = a \\ \lambda_c : e &\longmapsto c \\ a &\longmapsto ca = b \\ b &\longmapsto cb = a \\ c &\longmapsto e \end{aligned} \quad \begin{aligned} \lambda_e &= \begin{pmatrix} e & a & b & c \\ e & a & b & c \end{pmatrix} \\ \lambda_a &= \begin{pmatrix} e & a & b & c \\ a & e & c & b \end{pmatrix} \\ \lambda_b &= \begin{pmatrix} e & a & b & c \\ b & c & e & a \end{pmatrix} \\ \lambda_c &= \begin{pmatrix} e & a & b & c \\ c & b & a & e \end{pmatrix} \end{aligned} \quad \{\lambda_e, \lambda_a, \lambda_b, \lambda_c\} \subseteq \text{Sym}(G) \cong S_4$$

$$\begin{aligned} \lambda_e &\mapsto (1) = (2) = (3) = (4) \\ \lambda_a &\mapsto (1 \ 2)(3 \ 4) \\ \lambda_b &\mapsto (1 \ 3)(2 \ 4) \\ \lambda_c &\mapsto (1 \ 4)(2 \ 3) \end{aligned}$$

Let V be a vector space of dimension $|V_4| = 4$. Let $\{v_1, v_2, v_3, v_4\}$ be an ordered basis. Then

$$\begin{aligned} T_{\lambda_c} : V &\rightarrow V \\ v_i &\mapsto v_{\lambda_c(i)} = v_i \end{aligned}$$

and $[T_{\lambda_c}]_{\mathcal{B}} = I_4$ where $\mathcal{B} = \{e_1, e_2, e_3, e_4\}$ is the standard ordered basis. Similarly,

$$\begin{aligned} T_{\lambda_a} : V &\rightarrow V & T_{\lambda_b} : V &\rightarrow V & T_{\lambda_c} : V &\rightarrow V \\ v_i &\mapsto v_{\lambda_a(i)} & v_i &\mapsto v_{\lambda_b(i)} & v_i &\mapsto v_{\lambda_c(i)} \\ \begin{pmatrix} v_1 & v_2 & v_3 & v_4 \\ v_2 & v_1 & v_4 & v_3 \end{pmatrix} & & \begin{pmatrix} v_1 & v_2 & v_3 & v_4 \\ v_3 & v_4 & v_1 & v_2 \end{pmatrix} & & \begin{pmatrix} v_1 & v_2 & v_3 & v_4 \\ v_4 & v_3 & v_2 & v_1 \end{pmatrix} \end{aligned}$$

$$\text{Hence, } A = [T_{\lambda_a}]_{\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, B = [T_{\lambda_b}]_{\mathcal{B}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \text{ and } C = [T_{\lambda_c}]_{\mathcal{B}} =$$

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \text{ Hence, } V_4 \cong \{I_4, A, B, C\} \leq \text{GL}_4(F).$$

Ring representations

Definition 3: Representation of a ring

Suppose R is a ring. Then any ring homomorphism

$$\psi : R \rightarrow \text{End}(M)$$

is called a **representation** of the ring R ; here M is an (additive) abelian group, and $\text{End}(M)$ is the ring of group endomorphisms of M .

Exercise 4. Suppose $\psi : R \rightarrow \text{End}(M)$ is a ring homomorphism. Prove that

$$\begin{aligned} R \times M &\rightarrow M \\ (r, m) &\mapsto rm := \psi(r)(m) \end{aligned}$$

defines a left R -module M s.t. the induced ring homomorphism is ψ .

Proof. Task: Routine verification. ■

Question 3. Suppose R is a ring & M is an (additive) abelian group. Does there exist an action of R on M which makes M into an R -module? OR (same question) Does there exist a ring homomorphism $R \rightarrow \text{End}(M)$?

Every abelian group is a (unitary) $\mathbb{Z}$ -module in exactly one way

Consider the following problem from [Jac85, §3.2, p. 6, page 166]: Suppose $(M, +)$ is an abelian group. Prove that M can be made into a unitary $\mathbb{Z}$ -module in exactly one way. In other words, there is only one ring homomorphism $\mathbb{Z} \rightarrow \text{End}(M)$ which maps identity to identity.

In fact, there is only one ring homomorphism $\mathbb{Z} \rightarrow R$ (R a ring with identity 1) that maps 1 to 1. Let $f : \mathbb{Z} \rightarrow R$ (R a ring with identity 1) be a ring homomorphism s.t. $f(1) = 1$. Then, for any $n \in \mathbb{Z}$, $f(n) = nf(1)$. So,

$$\begin{aligned} \psi : \mathbb{Z} &\rightarrow \text{End}(M) \\ 1 &\mapsto \psi(1) : M \rightarrow M \\ a &\mapsto a \\ n &\mapsto \psi(n) : M \rightarrow M \\ a &\mapsto na \end{aligned}$$

Hence, any (additive) abelian group M can be made into a unitary $\mathbb{Z}$ -module in exactly one way.

1.2 Categories

Definition 4

A category $\mathcal{C}$ consists of a class $\text{ob}(\mathcal{C})$ of objects, usually denoted by $A, B, C, \dots$, & sets of morphisms (or arrows) between objects, usually denoted by $\text{Hom}(A, B)$ or $\text{Mor}(A, B)$ (where A, B are objects of $\mathcal{C}$) s.t. the following hold:

(i) for $A, B, C \in \text{ob}(\mathcal{C})$ there exists a function

$$\begin{aligned} \text{Hom}(A, B) \times \text{Hom}(B, C) &\longrightarrow \text{Hom}(A, C) \\ (f, g) &\longmapsto g \circ f \end{aligned}$$

satisfying $(f \circ g) \circ h = f \circ (g \circ h)$, wherever all are defined;

(ii) for each $A \in \text{ob}(\mathcal{C})$ there exists a distinguished element $1_A \in \text{Hom}(A, A)$ satisfying

$$f \circ 1_A = f \quad \& \quad 1_A \circ g = g \quad \forall f \in \text{Hom}(A, B), g \in \text{Hom}(C, A).$$

Exercise 5. Prove that 1_A , with the above property (ii), is unique for any $A \in \text{ob}(\mathcal{C})$.

We take a class of objects to avoid set-theoretic problems such as Russell's paradox. When the class $\text{ob}(\mathcal{C})$ is a set, $\mathcal{C}$ is called a *small category*. Our Definition 4 for a category is not entirely precise but it suffices for now. The following are *concrete categories*,

Example 1

Grp: objects are groups, morphisms are group homomorphisms between two groups.

Example 2

For a ring R , $R\text{-Mod}$ (or $\text{Mod-}R$): objects are left (or right) R -modules and morphisms are module homomorphisms.

Example 3

Set: objects are sets, morphisms are mappings between sets.

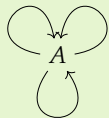
Example 4

Top: objects are topological spaces, morphisms are continuous mappings between topological spaces.

The following is an *abstract category*,

Example 5

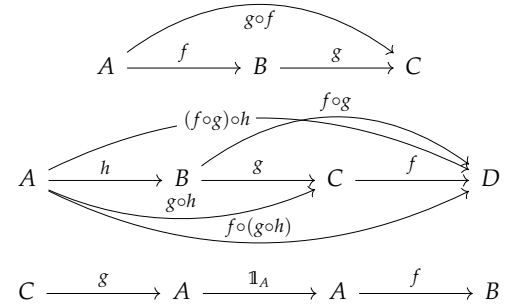
Mon: $\text{ob}(\text{Mon})$ consists of a single object A (say) and $1_A \in \text{Hom}(A, A)$. For all $f, g, h \in \text{Hom}(A, A)$, we have



$$f \circ (g \circ h) = (f \circ g) \circ h$$

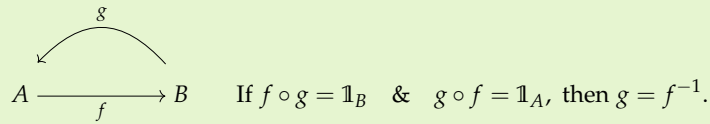
and $f \circ 1_A = 1_A \circ f = f$. We identify $1_A \equiv A$. So $\text{Hom}(A, A)$ is a monoid and this category **Mon** is called the *monoid category*. Conversely take a monoid $(M, \cdot)$ and identify 1_M with M then this becomes a category. Thus, a single object category is nothing but a monoid.

LECTURE 3
28th Jul, 2026



Inverse of an arrow/morphism in a category

Definition 5: Inverse of an arrow/morphism



Example 6

A one object category in which every arrow has an inverse is nothing but a group.

Example 7

A category $\mathcal{C}$, s.t. every arrow/morphism has an inverse, is called a *groupoid*.^a This is an example of an abstract category. A bundle of groups is a groupoid, but not the converse.

^aTo avoid confusion, we call a nonempty set equipped with a binary operation a *magma* instead of groupoid.

Counterexample 1

Let G be a directed graph, i.e. $G = (G^0, G^1, r, s)$ where G^0 is the set of vertices, G^1 is the set of edges, r is the range map and s is the source map; for $f \in G^1$

$$\begin{array}{ccc} r : G^1 \rightarrow G^0 & & \\ f \mapsto r(f) & & \\ s : G^1 \rightarrow G^0 & s(f) \bullet \xrightarrow{f} \bullet r(f) & \\ f \mapsto s(f) & & \end{array}$$

although this has similarities with a category as defined in Definition 4, it fails to be one. This is because the composition of two edges need not exist,

$$v_1 \bullet \xrightarrow{f} v_2 \bullet \xrightarrow{g} v_3 \bullet$$

just because there is an edge between v_1 and v_2 and an edge between v_2 and v_3 does not mean that there is an edge between v_1 and v_3 .

If we consider paths instead of edges as arrows, then we can make a directed graph into a category.

Example 8

Path category of a directed graph is an abstract category, where the objects are the vertices and morphisms are the paths. Identity is taken to be path of length 0, and every vertex is a path of length 0, every edge is a path of length 1. The composition of two paths is defined to be the concatenation of the two paths.

Correspondence between categories (functor)

We want to find a correspondence between the categories $R\text{-}\mathbf{Mod}$ and $\mathbf{Ab}$, where $R\text{-}\mathbf{Mod}$ is the category of left R -modules (morphisms are module homomorphisms) and $\mathbf{Ab}$ is the category of abelian groups (morphisms are group homomorphisms).

Let M, N be R -modules and $\text{Hom}(M, N) = \text{Hom}_R(M, N)$ = set of all R -module homomorphisms from M to N .

Recall: For vector spaces V, W over a field F , $\mathcal{L}(V, W) :=$ set of all F -linear maps from V to W is a vector space over F . For $T_1, T_2 \in \mathcal{L}(V, W)$ & $\lambda \in F$,

$$\begin{array}{ccc} T_1 + T_2 : V \longrightarrow W & & \lambda T_1 : V \longrightarrow W \\ v \longmapsto (T_1 + T_2)(v) := T_1(v) + T_2(v) & \& & v \longmapsto (\lambda T_1)(v) := \lambda T_1(v) \end{array}$$

Define for $f, g \in \text{Hom}(M, N)$ & $\lambda \in R$,

$$\begin{aligned} f + g : M &\longrightarrow N & \lambda f : M &\longrightarrow N \\ m &\longmapsto (f + g)(m) := f(m) + g(m) & m &\longmapsto (\lambda f)(m) := \lambda f(m) \end{aligned} \quad \&$$

Question 4. Is $\text{Hom}(M, N)$ a left R -module with the addition and multiplication defined above?

$f + g \in \text{Hom}(M, N)$ for all $f, g \in \text{Hom}(M, N)$ and $(\text{Hom}(M, N), +)$ is an abelian group. But for all $\lambda \in R$ & $f \in \text{Hom}(M, N)$, λf need not belong to $\text{Hom}(M, N)$.

$$\begin{aligned} (\lambda f)(m_1 + m_2) &= \lambda(f(m_1) + f(m_2)) \\ &= \lambda f(m_1) + \lambda f(m_2) \\ &= (\lambda f)(m_1) + (\lambda f)(m_2) \\ \text{but } (\lambda f)(\mu m_1) &= \lambda(f(\mu m_1)) \\ &= \lambda(\mu f(m_1)) \\ &\neq \mu(\lambda f(m_1)) \text{ as } R \text{ may not be commutative.} \end{aligned}$$

So all we can say is that $(\text{Hom}(M, N), +)$ is an abelian group.

Let M be an R -module.

$$\begin{aligned} R\text{-Mod} &\xrightarrow{\text{Hom}(M, -)} \mathbf{Ab} \\ N &\longmapsto \text{Hom}(M, N) \end{aligned}$$

$$\begin{array}{ccc} & & \text{Hom}(M, N_2) \\ & \nearrow & \uparrow \text{Hom}(M, f) \\ N_1 & \xrightarrow{f} & N_2 \\ & \searrow & \downarrow \text{Hom}(M, f) \\ & & \text{Hom}(M, N_1) \end{array} \quad \begin{array}{ccc} \text{Hom}(M, N_1) & \xrightarrow{\text{Hom}(M, f)} & \text{Hom}(M, N_2) \\ g & \longmapsto & f \circ g \\ (\text{Hom}(M, f))(g) & := & f \circ g \end{array}$$

$$\begin{array}{ccccc} & & & & \\ & & \text{Hom}(M, N_1) & \xrightarrow{\text{Hom}(M, f)} & \text{Hom}(M, N_2) \\ & & \searrow \text{Hom}(M, g \circ f) & & \downarrow \text{Hom}(M, g) \\ & & & & \text{Hom}(M, N_3) \end{array}$$

$$\begin{array}{ccccc} & & & & \\ & & \text{Hom}(M, N_1) & \xrightarrow{\text{Hom}(M, f)} & \text{Hom}(M, N_2) \\ & & \searrow \text{Hom}(M, g \circ f) & & \downarrow \text{Hom}(M, g) \\ & & & & \text{Hom}(M, N_3) \end{array}$$

$$\begin{array}{ccc} & M & \\ g \swarrow & & \searrow f \circ g \\ N_1 & \xrightarrow{f} & N_2 \end{array}$$

We decided what direction the arrow should be between $\text{Hom}(M, N_1)$ and $\text{Hom}(M, N_2)$ by looking at the directions of the arrows between N_1, N_2 and M . We also have an identity arrow $\mathbb{1}_{\text{Hom}(M, N_1)}$ on each object $\text{Hom}(M, N_1)$ in the category $\mathbf{Ab}$ given by

$$\begin{aligned} N_1 &\xrightarrow{\mathbb{1}_{N_1}} N_2 \\ \text{Hom}(M, N_1) &\xrightarrow{\text{Hom}(M, \mathbb{1}_{N_1})} \text{Hom}(M, N_2) \end{aligned} \quad \mathbb{1}_{\text{Hom}(M, N_1)} = \text{Hom}(M, \mathbb{1}_{N_1})$$

The above type of correspondence between categories is called a *functor*.

Precise definition of a category

We now give a precise definition of a category. Compare the following definition with Definition 4

LECTURE 4
31st Jul, 2026

Definition 6: Category

A **category** $\mathcal{C}$ consists of a class $\text{ob}(\mathcal{C})$ of *objects*, usually denoted by $A, B, C, \dots$ & a class $\text{Mor}(\mathcal{C})$ of *morphisms* (or *arrows*) between objects, whose elements are usually denoted by $f, g, \dots$ together with two mappings

$$\text{dom} : \text{Mor}(\mathcal{C}) \rightarrow \text{ob } \mathcal{C}$$

$$\text{cod} : \text{Mor}(\mathcal{C}) \rightarrow \text{ob } \mathcal{C}$$

$$\text{dom } f = A \xrightarrow{f} \text{cod } f = B \quad (\text{may be written as } f : A \rightarrow B)$$

& a mapping

$$\begin{aligned} \text{ob } \mathcal{C} &\longrightarrow \text{Mor}(\mathcal{C}) \\ A &\longmapsto \mathbb{1}_A : A \rightarrow A \end{aligned}$$

& s.t. there exists a *partial mapping/binary operation*

$$\begin{aligned} \text{Mor}(\mathcal{C}) \times \text{Mor}(\mathcal{C}) &\longrightarrow \text{Mor}(\mathcal{C}) \\ (f, g) &\longmapsto f \circ g \quad \text{iff } \text{cod } g = \text{dom } f \end{aligned}$$

satisfying

$$(f \circ g) \circ h = f \circ (g \circ h) \quad \forall f, g, h \in \text{Mor}(\mathcal{C}) \quad (1.2)$$

&

$$\mathbb{1}_A \circ f = f \text{ and } f \circ \mathbb{1}_A = f \quad \forall f \in \text{Mor}(\mathcal{C}) \quad (1.3)$$

whenever the compositions are defined.

If $\text{Mor}(\mathcal{C})$ is a set, then $\mathcal{C}$ is called a **locally small category**. If both $\text{ob}(\mathcal{C})$ & $\text{Mor}(\mathcal{C})$ are sets, then $\mathcal{C}$ is called a **small category**.

Category as a directed graph

So a category is a tuple $\mathcal{C} = (\text{ob}(\mathcal{C}) = V(\mathcal{C}), \text{Mor}(\mathcal{C}), \text{cod}, \text{dom}, \circ, \mathbb{1}_A)$

$$\text{cod} : \text{Mor}(\mathcal{C}) \rightarrow \text{ob}(\mathcal{C})$$

$$\text{dom} : \text{Mor}(\mathcal{C}) \rightarrow \text{ob}(\mathcal{C})$$

$$\text{ob}(\mathcal{C}) \rightarrow \text{Mor}(\mathcal{C})$$

$$A \mapsto \mathbb{1}_A : A \rightarrow A$$

$$\circ : \text{Mor}(\mathcal{C}) \times \text{Mor}(\mathcal{C}) \rightarrow \text{Mor}(\mathcal{C})$$

$$(f, g) \mapsto f \circ g \quad \text{iff } \text{cod } g = \text{dom } f$$

s.t. $(f \circ g) \circ h = f \circ (g \circ h)$ & $\mathbb{1}_A \circ f = f$ & $f \circ \mathbb{1}_A = f$ whenever the compositions are defined.

$$\text{dom } f \bullet \xrightarrow{f \in \text{Mor}(\mathcal{C})} \bullet \text{cod } f \quad f : \text{dom } f \rightarrow \text{cod } f$$

Recall: A directed graph is a tuple $E = (E^0, E^1, r, s)$ where $E^0 = v(E)$ is the set of vertices, E^1 is the set of edges, $r : E^1 \rightarrow E^0$ is the range map & $s : E^1 \rightarrow E^0$ is the source map. So a category is a directed graph with additional structure.

$$s(e) \bullet \xrightarrow{e} \bullet r(e) \quad \bullet \xrightarrow{e} \bullet \xrightarrow{f} \bullet$$

But this has no composition defined on it, since $\text{path} \neq \text{edge}$.

If we consider paths instead of edges as morphisms, vertex as path of length 0, edge as path of length 1, then directed graph becomes a category, called the *path category*.

$$\begin{aligned} s(e)e &= e \\ er(e) &= e \end{aligned} \quad \begin{aligned} &g \circ e \\ &\bullet \xrightarrow{e} \bullet \xrightarrow{f} \bullet \\ &\bullet \xrightarrow{ef} \bullet \end{aligned}$$

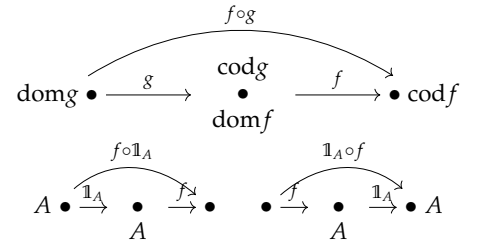


Figure 1.1: Diagram for Eq. (1.3)

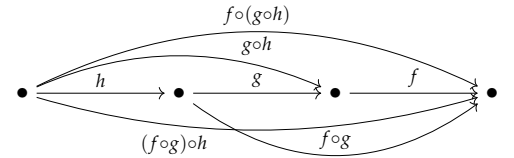


Figure 1.2: Diagram for Eq. (1.2)

Example 9

Equivalence class ρ on a set is a category.

$$\begin{array}{ccc} (a,b) \in \rho & & (c,d) \in \rho \\ a = \text{dom} f \xrightarrow{f} b = \text{cod} f & c = \text{dom} g \xrightarrow{g} d = \text{cod} g \end{array}$$

For $(a,b) \circ (c,d) = ?$ to make sense we must have $b = c$. So composition is defined as $(a,b) \circ (b,d) = (a,d)$. The identity morphism is (a,a) which exists due to reflexivity. Equivalence class is an example of a directed graph category with inverse arrows. Hence, equivalence class is a groupoid.

Warning. Taking Hasse diagrams in the above example won't work.

Example 10

Suppose $(P, \leq)$ is a pre-ordered set, i.e., $\leq$ is a pre-order on P (reflexive & transitive). Take

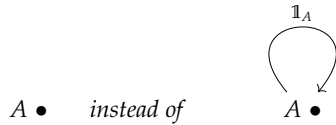
$$\begin{aligned} \text{ob}(\mathcal{C}) &:= P, \text{ Mor}(\mathcal{C}) = \{f : a \rightarrow b \mid a \leq b\} \\ &\text{for } a, b \in P, f \text{ is unique if exists} \end{aligned}$$

so $\text{Mor}(a,b) = \begin{cases} \{f\} \\ \emptyset \end{cases}$ or $\emptyset$. This is a category.
a singleton

In fact, any small category $\mathcal{C}$ s.t. for any two objects $A, B \in \text{ob}(\mathcal{C})$, $\text{Mor}(A,B)$ is either a singleton or empty, is nothing but a pre-ordered set.

Poset is also a category but due to antisymmetry no inverse arrows exist between distinct objects.

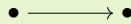
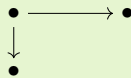
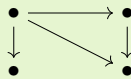
Convention 1. We don't draw the identity arrow loop, so we draw



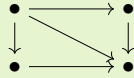
The following are some examples of categories,

Example 11

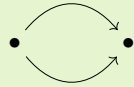
Discrete category i.e. only morphisms are the identity morphisms

**Example 12****Example 13****Example 14**

Example 15



Example 16

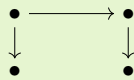


The following are *NOT* examples of categories,

Counterexample 2



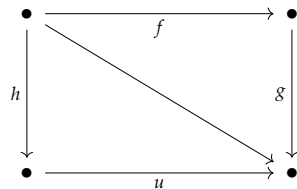
Counterexample 3



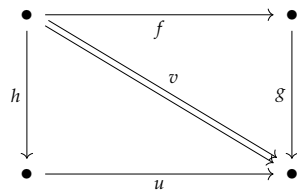
Counterexample 4



The diagram in Example 15 is an example of a *commutative diagram*

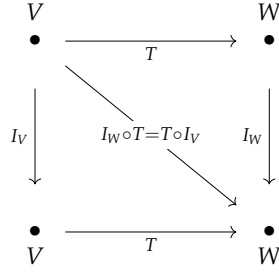


where the *commutativity* of a diagram of maps, when it makes sense, means that the maps obtained by following the diagram from one initial point to a terminal point along each displayed route are the same. The following diagram is not commutative, as there is another map $v \neq u \circ h = g \circ f$.

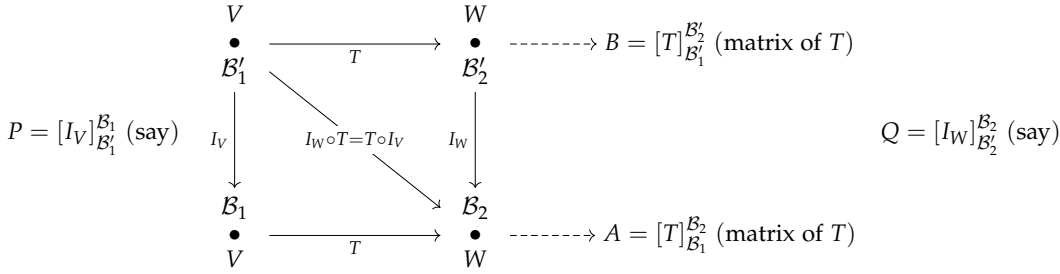


Derivation of the change-of-basis formula using commutative diagram

Consider two vector spaces V and W over F . Then we have the following commutative diagram



Let V and W be finite dimensional, consider the bases $\mathcal{B}_1, \mathcal{B}'_1$ of V and $\mathcal{B}_2, \mathcal{B}'_2$ of W .



$$\begin{aligned}
 \text{then } [I_W \circ T]_{\mathcal{B}'_1}^{\mathcal{B}_2} &= [T \circ I_V]_{\mathcal{B}'_1}^{\mathcal{B}_2} \implies [I_W]_{\mathcal{B}'_2}^{\mathcal{B}_2} \circ [T]_{\mathcal{B}'_1}^{\mathcal{B}'_2} = [T]_{\mathcal{B}_1}^{\mathcal{B}_2} \circ [I_V]_{\mathcal{B}'_1}^{\mathcal{B}_1} \\
 &\implies QB = AP \implies B = Q^{-1}AP \quad (\text{change-of-basis})
 \end{aligned}$$

Example 17

For any field K , $\mathbf{Vect}_K$ is a category.
 $\text{ob}(\mathbf{Vect}_K)$ Objects: all vector spaces over K
 $\text{Mor}(\mathbf{Vect}_K)$ Morphisms: all K -linear transformations

1.3 Functor

A *functor* is a correspondence between two categories satisfying some properties. To be precise,

Definition 7: Functor

Let $\mathcal{C}$ & $\mathcal{D}$ be two categories, then $F : \mathcal{C} \rightarrow \mathcal{D}$ is called a **functor** if it defines two functions^a

$$\begin{array}{ccc}
 \text{ob}(\mathcal{C}) & \xrightarrow{F} & \text{ob}(\mathcal{D}) \\
 A & \mapsto & FA
 \end{array}
 \quad \& \quad
 \begin{array}{ccc}
 \text{Mor}(\mathcal{C}) & \xrightarrow{F} & \text{Mor}(\mathcal{D}) \\
 f & \mapsto & Ff
 \end{array}$$

s.t.

- (i) for any $f \in \text{Mor}(\mathcal{C})$, $\text{dom}(Ff) = F(\text{dom } f)$ & $\text{cod}(Ff) = F(\text{cod } f)$ (i.e. F commutes with dom and cod)
- (ii) for any $A \in \text{ob}(\mathcal{C})$, $F1_A = 1_{FA}$
- (iii) for any $f, g \in \text{Mor}(\mathcal{C})$ if $g \circ f$ is defined in $\mathcal{C}$ then $Fg \circ Ff$ is defined in $\mathcal{D}$ and

$$F(g \circ f) = Fg \circ Ff.$$

^athere is some abuse of notation as we denote both the functions by the same letter F for convenience

Remark. Sometimes $F(g \circ f) = Ff \circ Fg$.

Remark. For any category $\mathcal{C}$, $\mathbb{1}_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$ is defined by

$$\begin{array}{ccc} \text{ob}(\mathcal{C}) & \xrightarrow{\mathbb{1}_{\mathcal{C}}} & \text{ob}(\mathcal{C}) \\ A & \longmapsto & A \end{array} \quad \& \quad \begin{array}{ccc} \text{Mor}(\mathcal{C}) & \xrightarrow{\mathbb{1}_{\mathcal{C}}} & \text{Mor}(\mathcal{C}) \\ f & \longmapsto & \mathbb{1}_{\mathcal{C}} f := f \end{array}$$

& it is called the *identity functor* of the category $\mathcal{C}$.

Contravariant and covariant functors

Recall: For any two modules M & N , $\text{Hom}(M, N)$ is an abelian group but not a module.

Let $M \in \text{ob}(R\text{-}\mathbf{Mod})$. Consider the correspondence $\text{Hom}(M, -)$ between $R\text{-}\mathbf{Mod}$ and $\mathbf{Ab}$,

$$R\text{-}\mathbf{Mod} \xrightarrow{\text{Hom}(M, -)} \mathbf{Ab}$$

$$\begin{array}{ccc} \text{ob}(R\text{-}\mathbf{Mod}) & \xrightarrow{\text{Hom}(M, -)} & \text{ob}(\mathbf{Ab}) \\ N & \longmapsto & \text{Hom}(M, N) \end{array}$$

$$\begin{array}{ccc} \text{Mor}(R\text{-}\mathbf{Mod}) & \xrightarrow{\text{Hom}(M, -)} & \text{Mor}(\mathbf{Ab}) \\ f & \longmapsto & \text{Hom}(M, f) : \text{Hom}(M, N_1) \rightarrow \text{Hom}(M, N_2) \\ & & g \mapsto \text{Hom}(M, f)(g) := ? \end{array}$$

$$\text{Ans. } ? = f \circ g.$$

Exercise 6. Prove that $\text{Hom}(M, -) : R\text{-}\mathbf{Mod} \rightarrow \mathbf{Ab}$ as defined above is a functor.

Let us now consider the correspondence $\text{Hom}(-, M)$ between $R\text{-}\mathbf{Mod}$ and $\mathbf{Ab}$,

$$R\text{-}\mathbf{Mod} \xrightarrow{\text{Hom}(-, M)} \mathbf{Ab}$$

defined by

$$\begin{array}{ccc} \text{ob}(R\text{-}\mathbf{Mod}) & \xrightarrow{\text{Hom}(-, M)} & \text{ob}(\mathbf{Ab}) \\ N & \longmapsto & \text{Hom}(N, M) \\ \text{Mor}(R\text{-}\mathbf{Mod}) & \xrightarrow{\text{Hom}(-, M)} & \text{Mor}(\mathbf{Ab}) \\ f & \longmapsto & \text{Hom}(f, M) : \text{Hom}(N_2, M) \rightarrow \text{Hom}(N_1, M) \\ & & g \mapsto \text{Hom}(f, M)(g) := g \circ f \end{array}$$

Notice how $\text{Hom}(-, M)$ reverses the directions of the arrows (*contravariance*) between M & N_1 and M and N_2 ; whilst $\text{Hom}(M, -)$ keeps the arrow directions same (*covariance*).

Definition 8: Contravariant and covariant functor

If a functor F keeps the directions of the arrows same, i.e. $F(f \circ g) = Ff \circ Fg$, then it is called a **covariant functor**. If, instead, a functor F reverses the directions of the arrows, i.e. $F(f \circ g) = Fg \circ Ff$, then it is called a **contravariant functor**.

Remark. Unless otherwise stated, covariant functor is simply written as functor.

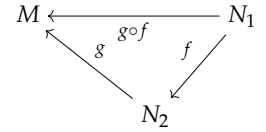
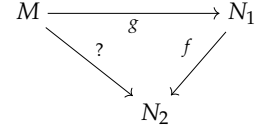
Exercise 7. Prove that $\text{Hom}(-, M)$ is a contravariant functor.

Example 18

$\mathbb{1}_{\mathbf{Vct}_K} : \mathbf{Vct}_K \rightarrow \mathbf{Vct}_K$, the identity functor is covariant.

Question 5. Is

$$\begin{array}{ccc} F : \mathbf{Vct}_K & \longrightarrow & \mathbf{Vct}_K \\ V & \longmapsto & V^{**} \end{array} \quad \text{a functor?}$$



Recall: $V^* = \mathcal{L}(V, K)$ is the dual space of V and $V^{**} = (V^*)^* = \mathcal{L}(V^*, K)$ is the double dual space of V . The linear transformation $f : V \rightarrow W$

$$V \bullet \xrightarrow{f} \bullet W$$

induces another linear transformation $f^* : W^* \rightarrow V^*$ in a canonical way, where $\mathcal{B}_1^*, \mathcal{B}_2^*$ are the dual bases of $\mathcal{B}_1, \mathcal{B}_2$.

The map $f^*(g) := g \circ f$ is called the *adjoint* or *transpose* of f . Indeed, $(f \circ g)^* = g^* \circ f^*$ since

$$(f \circ g)^*(h) = h \circ (f \circ g) = (h \circ f) \circ g = (f^*(h)) \circ g = g^*(f^*(h)) = (g^* \circ f^*)(h)$$

whenever the compositions are defined.

Question 6.

$$V^{**} \bullet \xrightarrow{Ff ?} \bullet W^{**}$$

What is the definition of Ff ?

By definition $Ff = f^{**} = (f^*)^*$ and

$$V^{**} \xrightarrow{f^{**}} W^{**}$$

$$(f^{**})(h) = (f^*)^*(h) = h \circ f^*$$

$$h \mapsto f^{**}(h)$$

so

$$f^{**}(h) : W^* \rightarrow K$$

$$g \mapsto (f^{**}(h))(g) = (h \circ f^*)(g) = h(f^*(g)) = h \circ (g \circ f)$$

&

$$h : V^* \rightarrow K$$

$$g \circ f \mapsto h \circ (g \circ f)$$

which is well-defined as $f : V \rightarrow W$ while $g : W \rightarrow K$ so $g \circ f : V \rightarrow K$, i.e., $g \circ f \in V^*$.

Also, there is a natural/canonical mapping

$$V \rightarrow V^{**}$$

$$v \mapsto \hat{v} = \phi_v : V^* \rightarrow K$$

$$f \mapsto \hat{v} = \phi_v(f) := f(v)$$

which is an example of a *natural transformation*.

1.4 Natural transformation

Definition 9: Natural transformation

Suppose $\mathcal{C}$ and $\mathcal{D}$ are two categories & $F, G : \mathcal{C} \rightarrow \mathcal{D}$ are two functors. A **natural transformation** $\alpha : F \rightarrow G$ is defined to be a function from $\text{ob}(\mathcal{C})$ to $\text{Mor}(\mathcal{D})$, denoted by

$$\alpha_A : FA \rightarrow GA \quad \forall A \in \text{ob}(\mathcal{C})$$

s.t. for any $f : A \rightarrow B$ in $\mathcal{C}$ the following diagram^a commutes

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ \alpha_A \downarrow & & \downarrow \alpha_B \\ GA & \xrightarrow{Gf} & GB \end{array} \quad \boxed{\alpha_B \circ Ff = Gf \circ \alpha_A}.$$

If each α_A is an isomorphism^b then α is called a **natural isomorphism**.

^athese diagrams are called *naturality squares* or *naturality diagrams*

^bi.e. α_A^{-1} exists

$$\begin{array}{ccc} W^* & & V^* \\ \bullet & \xrightarrow{\quad} & \bullet \\ \mathcal{B}_2^* & & \mathcal{B}_1^* \end{array}$$

$$W \xrightarrow{g} K \quad \xrightarrow{\quad} \quad V \xrightarrow{f^*(g)} K$$

Figure 1.3: $f^* : W^* \rightarrow V^*$

$$\begin{array}{ccc} W & & K \\ \bullet & \xrightarrow{g} & \bullet \\ \mathcal{B}_2 & & \\ \uparrow f & \nearrow f^*(g)=g \circ f & \\ V & & \\ \bullet & & \\ \mathcal{B}_1 & & \end{array}$$

LECTURE 5
4th Aug, 2026

Example 19

Let V be a vector space over a field K and V^{**} be its double dual space. We know $V^* = \mathcal{L}(V, K)$, $V^{**} = \mathcal{L}(V^*, K)$. We can define a canonical map $V \longrightarrow V^{**}$

$$\begin{aligned} V &\longrightarrow V^{**} \\ v &\longmapsto \hat{v} = \phi_v : V^* \longmapsto K \\ f &\longmapsto \phi_v(f) := f(v) \end{aligned}$$

which is meaningful as $f \in V^*$ so

$$\begin{aligned} f &: V \longrightarrow K \\ v &\longmapsto f(v) \end{aligned}$$

then

$$\begin{aligned} V &\longrightarrow V^{**} \\ v &\longmapsto \hat{v} \text{ or } \phi_v \end{aligned}$$

is a functor (why?).

Question 7. Let $G : \mathbf{Vect}_K \longrightarrow \mathbf{Vect}_K$ s.t.

$$\begin{aligned} V &\longmapsto V^{**} && \text{for } \text{ob}(\mathbf{Vect}_K) \\ f &\longmapsto f^{**} && \text{for } f \in \mathcal{L}(V, W), f^{**} \in \mathcal{L}(V^{**}, W^{**}) \end{aligned}$$

Is G a functor?

In this case we get the following naturality square,

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ \downarrow \alpha_V & & \downarrow \alpha_W \\ V^{**} & \xrightarrow{f^{**}} & W^{**} \end{array} \quad (1.4)$$

Functor category and Hom functors**Definition 10: Functor category**

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A **functor category**, denoted $[\mathcal{C}, \mathcal{D}]$ or $\mathcal{D}^{\mathcal{C}}$, is a category where $\text{ob}([\mathcal{C}, \mathcal{D}])$ is the class of all functors from $\mathcal{C}$ to $\mathcal{D}$ whilst $\text{Mor}([\mathcal{C}, \mathcal{D}])$ is the class of all natural transformations between two functors.

Exercise 8. Show that the functor category $[\mathcal{C}, \mathcal{D}]$ above is, indeed, a category.

Example 20

Let R be a ring. Consider the category $R\text{-}\mathbf{Mod}$ of left R -modules and the category $\mathbf{Ab}$ of abelian groups. Let $M \in \text{ob}(R\text{-}\mathbf{Mod})$. Define $\text{Hom}(M, -) : R\text{-}\mathbf{Mod} \longrightarrow \mathbf{Ab}$ by

$$\begin{aligned} \text{Hom}(M, -) &: \text{ob}(R\text{-}\mathbf{Mod}) \longrightarrow \text{ob}(\mathbf{Ab}) \\ N &\longmapsto \text{Hom}(M, N) \end{aligned}$$

$$\begin{aligned} \text{Hom}(M, -) &: \text{Mor}(R\text{-}\mathbf{Mod}) \longrightarrow \text{Mor}(\mathbf{Ab}) \\ \text{Hom}(N_1, N_2) \ni f &\longmapsto \text{Hom}(M, f) : \text{Hom}(M, N_1) \longrightarrow \text{Hom}(M, N_2) \\ g &\longmapsto \text{Hom}(M, f)(g) := f \circ g \end{aligned}$$

Exercise 9. Prove that $\text{Hom}(M, -)$ defined above is a functor. This is called a Hom functor.

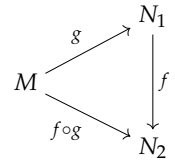


Figure 1.4: Composition in Example 20

Similarly, $\text{Hom}(-, M)$ is a contravariant Hom functor. Hom functors can be generalised to arbitrary categories.

Definition 11: Hom functor

Let $\mathcal{C}$ be any locally small category and $M \in \text{ob}(\mathcal{C})$. Then a **Hom functor** is defined to be a functor $\text{Hom}(M, -) : \mathcal{C} \longrightarrow \mathbf{Set}$ s.t.

$$\begin{aligned} \text{Hom}(M, -) : \text{ob}(\mathcal{C}) &\longrightarrow \text{ob}(\mathbf{Set}) \\ N &\longmapsto \text{Hom}(M, N) \\ \text{Hom}(M, -) : \text{Mor}(\mathcal{C}) &\longrightarrow \text{Mor}(\mathbf{Set}) \\ \text{Hom}(N_1, N_2) \ni f &\longmapsto \text{Hom}(M, f) := f \circ \\ &\text{Hom}(M, N_1) \longrightarrow \text{Hom}(M, N_2) \end{aligned}$$

and similarly $\text{Hom}(f, M) := \circ f$ for the *contravariant* case.

In this case we get the following naturality square,

$$\begin{array}{ccc} \text{Hom}(M, N) & \xrightarrow{\text{Hom}(M, f)} & \text{Hom}(M, N') \\ \text{Hom}(g, N) \downarrow & & \downarrow \text{Hom}(g, N') \\ \text{Hom}(M', N) & \xrightarrow{\text{Hom}(M', f)} & \text{Hom}(M', N') \end{array} \quad (1.5)$$

Use of functor

Consider the following sequence of modules and module homomorphisms in the category $R\text{-}\mathbf{Mod}$,

$$0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0 \quad (1.6)$$

which is said to be *exact* if the the image of each homomorphism is equal to the kernel of the next,

$$\text{im } 0 = \ker f, \text{ im } f = \ker g, \text{ im } g = \ker 0.$$

In particular, this sequence is an example of a *short exact sequence*. We have

$$\begin{aligned} \ker f = \text{im } 0 = \{0\} &\implies f \text{ is a monomorphism} \\ \text{im } g = \ker 0 = \{0\} &\implies g \text{ is an epimorphism} \\ M_3 \cong M_2 / \text{im } f = M_2 / \ker g, & \quad M_2 = M_1 \oplus M_3 \end{aligned}$$

Now apply $\text{Hom}(M, -)$ on the above short exact sequence (Eq. (1.6))

$$\text{Hom}(M, 0) \longrightarrow \text{Hom}(M, M_1) \xrightarrow{\text{Hom}(M, f)} \text{Hom}(M, M_2) \xrightarrow{\text{Hom}(M, g)} \text{Hom}(M, M_3) \longrightarrow \text{Hom}(M, 0)$$

is this sequence short exact as well? If yes, the functor $\text{Hom}(M, -)$ is *exact* and the module M is called a *projective module*. The $\text{Hom}(M, -)$ functor is always *left exact*¹, but if it is also *right exact*² then it is said to be an *exact functor*.

Hence, we get the very useful characterisation: a module $M \in \text{ob}(R\text{-}\mathbf{Mod})$ is projective if and only if the functor $\text{Hom}(M, -)$ is exact. Recall: Fix a field K . $\mathbb{1}_{\mathbf{Vect}_K} : \mathbf{Vect}_K \longrightarrow \mathbf{Vect}_K$ is the identity functor in the category of vector spaces over K ; further, we have the double dual functor

$$\begin{aligned} ()^{**} : \mathbf{Vect}_K &\longrightarrow \mathbf{Vect}_K \\ V &\longmapsto V^{**} \\ \mathcal{L}(V, W) \ni f &\longmapsto f^{**} \in \mathcal{L}(V^{**}, W^{**}) \end{aligned}$$

Then we have the natural transformation $\alpha : \mathbb{1}_{\mathbf{Vect}_K} \longrightarrow ()^{**}$ s.t. for $V \in \text{ob}(\mathbf{Vect}_K)$

$$\begin{aligned} V &\longmapsto \alpha_V : \mathbb{1}_{\mathbf{Vect}_K}(V) \longrightarrow (V)^{**} \\ \alpha_V : V &\longrightarrow V^{**} \\ v &\longmapsto \alpha_V(v) = \hat{v} \end{aligned}$$

¹i.e. $\text{Hom}(M, 0) \rightarrow \text{Hom}(M, M_1) \rightarrow \text{Hom}(M, M_2) \rightarrow \text{Hom}(M, M_3)$ is exact

²i.e. $\text{Hom}(M, M_1) \rightarrow \text{Hom}(M, M_2) \rightarrow \text{Hom}(M, M_3) \rightarrow \text{Hom}(M, 0)$ is exact

We want to show that the diagram Eq. (1.4) commutes, i.e., the following square

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ \alpha_V \downarrow & & \downarrow \alpha_W \\ V^{**} & \xrightarrow{f^{**}} & W^{**} \end{array}$$

is indeed a naturality square; in other words, we have to show that

$$\boxed{f^{**} \circ \alpha_V = \alpha_W \circ f}$$

Let $v \in V$. Then

$$\begin{aligned} (\alpha_W \circ f)(v) &= \alpha_W(f(v)) = \widehat{f(v)} \in W^{**} \\ (f^{**} \circ \alpha_V)(v) &= f^{**}(\alpha_V(v)) = f^{**}(\hat{v}) = \hat{v} \circ f^* \\ \widehat{f(v)} : W^* &\longrightarrow K \\ g &\longmapsto \widehat{f(v)}(g) = g(f(v)) \end{aligned}$$

$$\begin{aligned} \hat{v} \circ f^* : W^* &\longrightarrow K \\ g &\longmapsto (\hat{v} \circ f^*)(g) = \hat{v}(f^*(g)) \\ &= \hat{v}(g \circ f) \\ &= (g \circ f)(v) \\ &= g(f(v)) \end{aligned}$$

and as $g \in W^*$ was arbitrary, we have $(\alpha_W \circ f)(v) = \widehat{f(v)} = \hat{v} \circ f^* = (f^{**} \circ \alpha_V)(v)$ for all $v \in V$.

Hence the diagram commutes and $\alpha : \mathbb{1} \longrightarrow (\)^{**}$ is a natural transformation. This is also why

$$\begin{aligned} \alpha_V : V &\longrightarrow V^{**} \\ v &\longmapsto \hat{v} \end{aligned}$$

is called a *natural/canonical map*.

1.5 Equivalent and isomorphic categories, opposite category

Definition 12: Equivalent and isomorphic categories

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and

$$F : \mathcal{C} \rightarrow \mathcal{D}, \quad G : \mathcal{D} \rightarrow \mathcal{C}$$

be two functors. If there is a natural isomorphism

$$\eta : \mathbb{1}_{\mathcal{C}} \rightarrow GF$$

& a natural isomorphism

$$\zeta : FG \rightarrow \mathbb{1}_{\mathcal{D}}$$

then we write $\mathcal{C} \cong \mathcal{D}$ (or $\mathcal{C} \simeq \mathcal{D}$) and say that $\mathcal{C}$ and $\mathcal{D}$ are **equivalent categories**.

If $GF = \mathbb{1}_{\mathcal{C}}$ and $FG = \mathbb{1}_{\mathcal{D}}$ then $\mathcal{C}$ and $\mathcal{D}$ are said to be **isomorphic categories**.

Definition 13: Opposite category

If $\mathcal{C}$ is a category, then $\mathcal{C}^{\text{op}}$ with $\text{ob}(\mathcal{C}^{\text{op}}) = \text{ob}(\mathcal{C})$ &

$$\text{Mor}_{\mathcal{C}^{\text{op}}}(A, B) = \text{Mor}_{\mathcal{C}}(B, A)$$

is called the **opposite category** of $\mathcal{C}$.

1.6 Adjunction

Definition 14: Left adjoint

Let $\mathcal{C}$ & $\mathcal{D}$ be two categories and

$$F : \mathcal{C} \rightarrow \mathcal{D}, \quad G : \mathcal{D} \rightarrow \mathcal{C}$$

be two functors. Then F is said to be a **left adjoint** of G , written as $F \dashv G$, if for any $A \in \text{ob}(\mathcal{C})$ & for any $B \in \text{ob}(\mathcal{D})$ there is a bijection between

$$\text{Mor}_{\mathcal{D}}(FA, B) \quad \& \quad \text{Mor}_{\mathcal{C}}(A, GB).$$

Note. When F is a left adjoint of G , G is a right adjoint of F .

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ & & \\ A & \xrightarrow{\quad} & FA \\ \text{\scriptsize (dashed ellipse)} & & \\ \mathcal{C} & \xrightarrow{G} & \mathcal{D} \\ & & \\ B & \xrightarrow{\quad} & GB \end{array}$$

Question 8. Suppose $\mathcal{C}$ and $\mathcal{D}$ are equivalent categories,

$$\begin{array}{ccc} \mathcal{C} & \xrightleftharpoons[\simeq G]{F} & \mathcal{D} \\ A & & B \end{array} \quad \text{then do we have} \quad \text{Mor}(FA, B) \xleftarrow{?} \text{Mor}(A, GB)$$

i.e. is there a bijection between $\text{Mor}(FA, B)$ and $\text{Mor}(A, GB)$?

From the definition of equivalent categories, there are two natural isomorphisms

$$\begin{aligned} \alpha : \mathbb{1}_{\mathcal{C}} &\longrightarrow GF, & \beta : FG &\longrightarrow \mathbb{1}_{\mathcal{D}} \\ \alpha_A : A &\longrightarrow GFA, & \beta_B : FGB &\longrightarrow B \end{aligned}$$

where $A \in \text{ob}(\mathcal{C})$ and $B \in \text{ob}(\mathcal{D})$.

$$\text{Mor}(FA, B) \quad \text{Mor}(A, GB)$$

$$f \xrightarrow{\psi} Gf \circ \alpha_A$$

$$\beta_B \circ Fh \xleftarrow{\phi} h$$

We first show injectivity. Let $\psi(f) = Gf \circ \alpha_A = Gg \circ \alpha_A = \psi(g)$. Then using “right cancellation” we get

$$\begin{aligned} Gf &= Gg \\ FGf &= FGg \end{aligned} \tag{1.7}$$

We then have the following diagram

$$\begin{array}{ccc} FGFA & \xrightarrow[\quad FGf]{\quad FGg \quad} & FGB \\ \beta_{FA} \downarrow & & \downarrow \beta_B \\ FA & \xrightarrow[\quad f \quad]{\quad g \quad} & B \end{array}$$

whence we get

$$\begin{aligned} \beta_B \circ FGf &= f \circ \beta_{FA} \\ \beta_B \circ FGg &= g \circ \beta_{FA} \end{aligned} \tag{1.8}$$

then, since $\beta_B \circ FGf = \beta_B \circ FGg$, we get

$$\begin{aligned} f \circ \beta_{FA} &= g \circ \beta_{FA} \\ \implies f &= g \end{aligned}$$

where we “right cancel” once again. Hence ψ is injective. For surjectivity, take an $h \in \text{Mor}(A, GB)$

$$\begin{array}{ccc} A & \xrightarrow{h} & GB \\ \\ FA & \xrightarrow{Fh} & FGB \\ \downarrow ? & \swarrow \beta_B & \\ B & & \end{array} \quad \text{Ans. } ? = \beta_B \circ Fh.$$

$$\begin{array}{ccc} GFA & \xrightarrow{Gf} & GB \\ \uparrow \alpha_A & \nearrow Gf \circ \alpha_A & \\ A & & \end{array} \quad \begin{aligned} (\phi \circ \psi)(f) & \\ &= \phi(Gf \circ \alpha_A) \\ &= \beta_B \circ (F(Gf \circ \alpha_A)) \\ &= \beta_B \circ (FGf \circ F\alpha_A) \end{aligned} \quad \begin{array}{ccc} FA & \xrightarrow{F(Gf \circ \alpha_A)} & FGB \\ & \searrow \beta_B \circ (F(Gf \circ \alpha_A)) & \downarrow \beta_B \\ & & B \end{array}$$

$\beta_B \circ (F(Gf \circ \alpha_A)) = (\phi \circ \psi)(f) \stackrel{?}{=} f$

Using Eq. (1.8), $\beta_B \circ (FGf \circ F\alpha_A) = (\beta_B \circ FGf) \circ F\alpha_A = (f \circ \beta_{FA}) \circ F\alpha_A = f \circ (\beta_{FA} \circ F\alpha_A)$

2. Unitary Modules

LECTURE 1
17th Jul, 2026

We will assume all rings to have identity unless otherwise stated. Recall the definition of a unitary left R -module: given $M \neq \emptyset$, $(M, +)$ an abelian group, $R, +, \cdot$ a ring and a binary operation

$$\begin{aligned} \cdot : R \times M &\longrightarrow M \\ (r, m) &\longmapsto rm \end{aligned}$$

satisfying the following properties:

- (i) $r(m_1 + m_2) = rm_1 + rm_2$
- (ii) $(r_1 + r_2)m = r_1m + r_2m$
- (iii) $(r_1r_2)m = r_1(r_2m)$
- (iv) $1_Rm = m$ (unitary module)

for $r, r_1, r_2 \in R$ and $m, m_1, m_2 \in M$; then M is said to be a unitary left R -module. We will assume all modules to be unitary unless otherwise stated. Note that left and right R -modules are not the same unless R is commutative. If R is not commutative, we will by default use R -module to mean *left* R -module.

Example 21

- (i) Every vector space over a field F is an F -module.
- (ii) Every additive abelian group is a $\mathbb{Z}$ -module in *exactly one way* as we've seen.
- (iii) Every ring R itself is an R -module.

2.1 Submodules

A lot of the theory for vector spaces and modules is similar, until we get to anything that requires commutativity or division by nonzero elements. Exactly analogous to the notion of a subspace of a vector space, we have the notion of a submodule:

Definition 15: Submodule

A nonempty subset $S \subseteq M$ of an R -module M is a **submodule** of M if S itself is an R -module. Equivalently, $\emptyset \neq S \subseteq M$ is a submodule of M if for all $r \in R$ and $x, y \in S$ we have $x + ry \in S$.

Example 22

For any ring R the submodules of R considered as a module over itself are the ideals I of R . For example, all the submodules of $\mathbb{Z}$ are $n\mathbb{Z}$.

Notation 1. We write $S \leq M$ to mean that S is a submodule of M .

Recall that the sum and intersections of subspaces is also a subspace, but the union of subspaces need not be a subspace. We have similar results for submodules:

Theorem 2

Let M be an R -module and $S_1, S_2 \leq M$. Define $S_1 + S_2 = \{s_1 + s_2 : s_1 \in S_1, s_2 \in S_2\}$. Then $S_1 + S_2, S_1 \cap S_2 \leq M$ but $S_1 \cup S_2 \not\leq M$.

Proof. As $S_1, S_2 \leq M$, clearly $0 \in S_1, S_2$ so $0 = 0 + 0 \in S_1 + S_2$ and $0 \in S_1 \cap S_2$. Let $r \in R$. Then

$$s_1 + s_2, s'_1 + s'_2 \in S_1 + S_2 \implies s_1 + s_2 + r(s'_1 + s'_2) = (s_1 + rs'_1) + (s_2 + rs'_2) \in S_1 + S_2$$

as $s_1 + rs'_1 \in S_1$ and $s_2 + rs'_2 \in S_2$ due to $S_1, S_2 \leq M$; also,

$$s, s' \in S_1 \cap S_2 \implies s, s' \in S_1, S_2 \implies s + rs' \in S_1, S_2 \implies s + rs' \in S_1 \cap S_2.$$

Hence, $S_1 + S_2, S_1 \cap S_2 \leq M$. Now, consider the $\mathbb{Z}$ -module $\mathbb{Z}$. Then $2\mathbb{Z}, 3\mathbb{Z} \leq \mathbb{Z}$ but $2\mathbb{Z} \cup 3\mathbb{Z} \not\leq \mathbb{Z}$; take $2 \in 2\mathbb{Z}, 3 \in 3\mathbb{Z}$ then $2 + 3 = 5 \notin 2\mathbb{Z} \cup 3\mathbb{Z}$. ■

Every vector space has a direct sum decomposition: both trivial ($U \oplus \{0\} = V$) and non-trivial ($U \oplus U^\perp = V$.) However, the same is not true for modules. Consider again the $\mathbb{Z}$ -module $\mathbb{Z}$ and if possible let $\mathbb{Z} = n\mathbb{Z} \oplus m\mathbb{Z}$. For any two algebraic structures A, B we have $A \oplus B \cong A \times B$, by the map

$$\begin{aligned} f : A \oplus B &\longrightarrow A \times B \\ a + b &\longmapsto (a, b) \end{aligned}$$

whence, taking $G = A \oplus B \cong A \times B$, one gets $G/A \cong B$. So from our assumption,

$$\mathbb{Z} = n\mathbb{Z} \oplus m\mathbb{Z} \cong n\mathbb{Z} \times m\mathbb{Z} \implies \mathbb{Z}/n\mathbb{Z} \cong m\mathbb{Z}$$

which is absurd as $\mathbb{Z}/n\mathbb{Z}$ is finite so it cannot be isomorphic to the infinite $m\mathbb{Z}$.

2.2 Free modules

The following definitions are analogous to that of vector spaces or even groups,

LECTURE 2
22nd Jul, 2026

Definition 16: Submodules generated by a set

Let M be an R -module and $S \neq \emptyset$ be any nonempty set. The **submodule $\langle S \rangle$ generated by S^a** is the smallest submodule of M containing S , i.e.,

$$\text{span}(S) = \langle S \rangle = \bigcap_{S \subseteq N_S \leq M} N_S.$$

If $S = \{x_1, \dots, x_n\}$ is finite, then we write $\langle x_1, \dots, x_n \rangle = \langle \{x_1, \dots, x_n\} \rangle$ and call the submodule *finitely generated*. In particular,

$$\langle x_1, \dots, x_n \rangle = Rx_1 + \dots + Rx_n.$$

For $n = 1$, $\langle x_1 \rangle = Rx_1$ is called the *cyclic submodule* generated by x_1 .

^aor, from vector space terminology, the *span* $\text{span}(S)$ of S

Definition 17: Linear independence

A nonempty subset S of an R -module M is said to be **R -linearly independent** (or simply *linearly independent*) if for all possible finite subsets $\{x_1, \dots, x_n\}$ of S we have

$$r_1x_1 + \dots + r_nx_n = 0 \implies r_1 = \dots = r_n = 0 \quad \forall r_1, \dots, r_n \in R.$$

Definition 18: Basis

A subset $\mathcal{B}$ of an R -module M is an **R -basis** (or simply *basis*) of M if $\mathcal{B}$ is *linearly independent* and $\text{span } \mathcal{B} = \langle \mathcal{B} \rangle = M$.

Recall that for a vector space V over a field F the following are equivalent:

- (1) $\mathcal{B}$ is an F -basis.
- (2) $\mathcal{B}$ is a maximal linearly independent subset of V .
- (3) $\mathcal{B}$ is a minimal spanning set of V .

Is the basis ever unique for a (non-trivial) vector space? A group G has a unique generator iff $|G| = 2$. Similarly, V has a unique F -basis iff $\dim_F(V) = 1, |F| = 2$. For a vector space (group) we can always find a basis (set of generators), and the same is true for its subspaces (subgroups). However, the same is not true for modules - and even if a basis $\mathcal{B}$ *does exist*, we can sometimes find another basis $\mathcal{B}'$ s.t. $|\mathcal{B}| \neq |\mathcal{B}'|$!

The $\mathbb{Z}$ -module $\mathbb{Z}$ has the bases $\{1\}$ and $\{-1\}$ but the $\mathbb{Z}$ -modules $\mathbb{Q}$ and $\mathbb{Z}/(n)$ don't have any basis. We say a module M is a *free module* if it has a basis. Any ring R over itself is a free module. If for every finitely generated module over R , the bases have the same cardinality, then R is said to have the *invariant basis number (IBN)* property.

$\mathbb{Z} \times \mathbb{Z}$ is a free module over itself but $\mathbb{Z} \times \{0\}$ is not a free module over $\mathbb{Z} \times \mathbb{Z}$.

Theorem 3: Modular law

If M is an R -module and $A, B, C \leq M$ with $C \subseteq A$, then

$$A \cap (B + C) = (A \cap B) + C.$$

Proof. $C \subseteq A$ so $A \cap C = C$. Also, $A \cap B \subseteq A, B$. Now,

$$(A \cap B) + C \subseteq A + C \subseteq A + A \subseteq A,$$

$$(A \cap B) + C \subseteq A + C \subseteq B + C.$$

Hence, $(A \cap B) + C \subseteq A \cap (B + C)$. Now let $x \in A \cap (B + C)$. Then $x \in A$ and $x \in B + C$,

$$\implies x = b + c, b \in B, c \in C \implies x - c = b \in B.$$

But $c \in C \subseteq A$, so $x - c \in A$. Hence, $x = x - c + c \in (A \cap B) + C$. ■

Exercise 10. (1) Show that the $\mathbb{Z}$ -module $\mathbb{Q}$ has no finite set of generators.

(2) Show that the $\mathbb{Z}$ -module $\mathbb{Q}$ cannot be expressed as a direct sum of two nonzero submodules of $\mathbb{Q}$.

(3) If e is a central idempotent in a ring R and M is an R -module, then show that

$$M = eM \oplus (1 - e)M.$$

Solution. (1) Suppose for contradiction that there is a finite set of generators $\left\{ \frac{p_1}{q_1}, \dots, \frac{p_n}{q_n} \right\}$ with $\gcd(p_i, q_i) = 1, q_i \neq 0$. Then

$$r_1 \frac{p_1}{q_1} + \dots + r_n \frac{p_n}{q_n} = \frac{p}{q} \quad \forall \frac{p}{q} \in \mathbb{Q}.$$

Now there exists a prime p s.t. p does not divide $q_1, \dots, q_n$; clearly, as there are infinitely many primes. Let

$$r_1 \frac{p_1}{q_1} + \dots + r_n \frac{p_n}{q_n} = \frac{p}{q} \quad \forall \frac{1}{p},$$

then

$$r_1 p_1 + \dots + r_n p_n = \frac{q_1 \cdots q_n}{p}$$

where the LHS is an integer, so the RHS must be as well but then p divides $q_1 \cdots q_n$ so p divides at least one of $q_1, \dots, q_n$; contradiction.

(2) Let $\mathbb{Q} = \mathbb{Q}_1 \oplus \mathbb{Q}_2$ then $\mathbb{Q} = \mathbb{Q}_1 + \mathbb{Q}_2$ with $\mathbb{Q}_1 \cap \mathbb{Q}_2 = \{0\}$. Let

$$q_1 = \frac{a_1}{b_1} \in A \setminus \{0\}, q_2 = \frac{a_2}{b_2} \in B \setminus \{0\}$$

then $b_1 a_2 q_1 \in A$ and $b_2 a_1 q_2 \in B$. Now,

$$b_1 a_2 q_1 = b_1 a_2 \frac{a_1}{b_1} = a_2 a_1,$$

$$b_2 a_1 q_2 = b_2 a_1 \frac{a_2}{b_2} = a_1 a_2$$

so $b_1 a_2 q_1 = a_1 a_2 = b_2 a_1 q_2 \in A \cap B$. Now, $a_1, a_2 \neq 0$ but $A \cap B = \{0\}$; contradiction.

(3) Let e be a central idempotent i.e. $e^2 = e$ and $er = re$ for all $r \in R$. Let M be an R -module, then

$$eM = \{em : m \in M\}, \quad (1 - e)M = \{(1 - e)m : m \in M\}.$$

It is easy to see that $eM, (1 - e)M \leq M$. Clearly, $M = eM + (1 - e)M$ as $m = em + (1 - e)m \in eM + (1 - e)M$ for any $m \in M$. It remains to show that the sum is direct. Clearly $\{0\} \subseteq eM \cap (1 - e)M$.

Now let $m \in eM \cap (1-e)M$, then $m = em_1$ and $m = (1-e)m_2$ for some $m_1, m_2 \in M$. So,

$$em = e(1-e)m_2 = (e-e^2)m_2 = (e-e)m_2 = 0m_2 = 0$$

so $em = 0$. Further, $m = em_1 \implies em = e^2m_1 = em_1 \implies em_1 = em = 0$. But then $m = em_1 = 0$, so $eM \cap (1-e)M = \{0\}$ and hence the sum is direct. ■

LECTURE 3
5th Aug, 2026

2.3 Quotient modules and isomorphisms

For an algebraic structure, it makes sense to ask whether we can study mappings on that structure that preserve said structure i.e. *homomorphisms*. The study of homomorphisms naturally leads to *quotient* structures and *isomorphic* structures. We have already done this for groups, rings and vector spaces. We simply generalise those familiar notions to that of a module.

Definition 19: Quotient module

Given a ring R , module M and submodule $N \leq M$, we define the **quotient module** M/N as

$$M/N = \{a + N : a \in M\}$$

s.t.

- (1) $(a + N) + (b + N) = (a + b) + N$, and
- (2) $(a + N)(b + N) = ab + N$.

A quotient module always exists: the natural example is $M/\ker f$ for any homomorphism f on M . In particular, if f is an epimorphism then we know that $M/\ker f \cong \text{im } f$.

Theorem 4: Isomorphism theorems

Let R be a ring and M a module.

- (1) $M/\ker f \cong \text{im } f$ if $f : M \rightarrow N$ is an epimorphism.
- (2) $N_1, N_2 \leq M$ then $N_1 / (N_1 \cap N_2) \cong (N_1 + N_2) / N_2$.
- (3) $A, B \leq M$ s.t. $A \subseteq B$ then $(M/A) / (B/A) \cong M/B$.

A **module homomorphism** is simply a map f that preserves the module properties:

$$f(a + rb) = f(a) + rf(b) \quad \forall a, b \in M, r \in R.$$

Further, f is a *monomorphism* if injective, an *epimorphism* if surjective, and an *isomorphism* if both i.e. bijective. The *image* of f is

$$\text{im } f = f(M) = \{f(m) : m \in M\}$$

and the *kernel* of f is

$$\ker f = \{m \in M : f(m) = 0\}.$$

Then $\ker f \leq M$ and $\text{im } f \leq N$.

2.4 Simple modules

Definition 20: Simple module

$M \neq \{0\}$ is said to be a **simple module** if the only submodules of M are $\{0\}$ and M itself.

Proposition 1

Every commutative simple $\mathbb{Z}$ -module is isomorphic to $\mathbb{Z}/p\mathbb{Z}$.

Lemma 1: Schur's Lemma

If M is a simple R -module then $\text{End}_R(M)$ is a division ring.

Simple group: $G \neq \{e\}$ s.t.

$$H \trianglelefteq G \implies H = \{e\}, G.$$

Simple ring: $R^2 \neq \{0\}$ s.t.

$$I \trianglelefteq R \implies I = \{0\}, R.$$

Every simple abelian group is isomorphic to $\mathbb{Z}/p\mathbb{Z}$.
Every commutative simple ring is a field.

Proof. Let $f \in \text{End}_R(M)$ s.t. $f \neq 0$. Then $\text{im } f, \ker f \leq M$. As M is simple, this implies $\text{im } f = \{0\}$ or M , $\ker f = \{0\}$ or M . If $\ker f = \{0\}$ (injective) and $\text{im } f = M$ (surjective) then we're done. If $\text{im } f = \{0\}$ then f maps every element to 0 i.e. $f = 0$ which is absurd. If $\ker f = M$, then by the same argument $f = 0$ which is absurd. ■

Definition 21: Semisimple module

M is said to be a **semisimple module** (or *completely reducible*) if it is the direct sum of *simple* (irreducible) modules.

Definition 22: Regular ring [Neu36, von Neumann, 1936]

A ring R is said to be a **von Neumann regular ring** or simply a **regular ring** if for all $a \in R$ there exists an $x \in R$ s.t. $\boxed{axa = a}$.

Corollary 1: Corollary to Schur's Lemma 1

If M is a semisimple R -module then $\text{End}_R(M)$ is a regular ring.

Exercise 11. 1. Let M be an R module and $f \in \text{End}_R(M)$ s.t. $f^2 = f$. Show that

$$\boxed{\ker f \oplus \text{im } f = M}.$$

2. Let R be an integral domain. Given $x \in R$ with $x \neq 0$, show that there is a descending chain

$$R \supseteq Rx \supseteq Rx^2 \supseteq \cdots \supseteq Rx^n \supseteq Rx^{n+1} \supseteq \cdots$$

of submodules of the R -module R s.t. for every n , $Rx^n / Rx^{n+1} \cong R / Rx$.

3. Show that every finitely generated R -module is a homomorphic image of a finitely generated free R -module.

4. If M is a cyclic R -module, then show that $M \cong R/I$, where I is a left ideal of R . If M is simple, then show that $M \cong R/\mathfrak{m}$, where $\mathfrak{m}$ is a maximal left ideal of R .

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Bibliography

- [Jac85] Nathan Jacobson. *Basic Algebra 1*. W. H. Freeman and Company, 1985.
- [Neu36] John von Neumann. "On Regular Rings". In: *Proceedings of the National Academy of Sciences* 22.12 (1936), pp. 707–713. doi: [10.1073/pnas.22.12.707](https://doi.org/10.1073/pnas.22.12.707). eprint: <https://www.pnas.org/doi/pdf/10.1073/pnas.22.12.707>. URL: <https://www.pnas.org/doi/abs/10.1073/pnas.22.12.707>.